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Technical Report No. 5

LINEAR STABILITY OF A TWO-PHASE PROCESS

INVOLVING A STEADILY PROPAGATING

PLANAR PHASE BOUNDARY IN A SOLID:

PART 2. THERMOMECHANICAL CASE

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ABSTRACT

This investigation is directed toward understanding the role of coupled mechanical and thermal effects in the linear stability of an isothermal antiplane shear motion which involves a single planar phase boundary in a *non-elliptic* thermoelastic material which has multiple elliptic phases. When the relevant process is static—so that the phase boundary does not move prior to the imposition of the disturbance—it is shown to be linearly stable. However, when the process involves a steadily propagating phase boundary it may be linearly unstable. Various conditions sufficient to guarantee the linear instability of the process are obtained. These conditions depend on the monotonicity of the *kinetic response function*—a constitutively supplied entity which relates the *driving traction* acting on a phase boundary to the local absolute temperature and the normal velocity of the phase boundary—and, in certain cases, on the spectrum of wave-numbers associated with the perturbation to which the process is subjected. Inertia is found to play an insignificant role in the qualitative features of the aforementioned sufficient conditions. It is shown, in particular, that instability can arise even when the normal velocity of the phase boundary is an increasing function of the driving traction if the temperature dependence in the kinetic response function is of a suitable nature. Significantly, the instability which is present in this setting occurs only in the *long waves* of the Fourier decomposition of the moving phase boundary, implying that the interface prefers to be highly wrinkled.

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1. INTRODUCTION

Recently [9], motivated by a desire to determine whether continuum mechanical models for displacive solid-solid phase transformations can predict the emergence of *plate-like* or *dendritic* structures from states involving planar phase boundaries, a purely mechanical two-phase dynamical process in a *non-elliptic* generalized neo-Hookean material was considered. The process involved an antiplane shear motion with a single steadily propagating planar *phase boundary* separating *high* and *low* strain elliptic phases of the relevant material. In a frame moving with the phase boundary, the shear strain field was piecewise homogeneous and the angle between the limiting values of the gradient of the out-of-plane displacement field on either side of the phase boundary was zero—so that the phase boundary was, for each instant of the motion, of *normal* type. The linear stability of this process with respect to a broad class of perturbations was then investigated. It was shown that a necessary and sufficient condition for the process to be linearly stable was that the kinetic response function—which gives the *driving traction* acting on a phase boundary in terms of the normal velocity of the phase boundary, or *vice-versa*—be a locally increasing function of its argument at the value corresponding to the base process. A necessary consequence of this stability criterion is that, in order for the process to be unstable, the kinetic response function must exhibit a non-monotonic dependence on its argument. Non-monotonic kinetic response functions are admissible under the *Clausius-Duhem* version of the second law of thermodynamics (specialized to isothermal conditions for the purposes of the purely mechanical process discussed in [9]); the work of OWEN, SCHOEN & SRINIVASAN [15] implies, furthermore, that a non-monotonic relation between interfacial driving traction and normal velocity may be operative in the *unstable kinetics* which are observed to accompany the emergence and growth of plate-like structures. Under such kinetics, the results obtained [9] suggest than an evolution from a planar to a plate-like phase boundary morphology might be possible with the confines of a purely mechanical theory.

Thermal effects are manifestly absent from the purely mechanical investigation in [9]. The experimental work of CLAPP & YU [5], GRUJICIC, OLSON & OWEN [10] and CONG DAHN, MORPHY & RAJAN [6] indicates that temperature effects do play an intrinsic, if not entirely understood, part in the kinetics of phase boundaries in displacive solid-solid phase transformations. The investigation which follows is, therefore, directed toward understanding the outcome, with regard to the morphological stability of states involving planar phase boundaries, when thermal as well as mechanical effects are taken into consideration in a model for displacive solid-solid phase transformations. Of particular interest is the question of whether thermal effects allow for an evolution from planar to plate-like phase boundary morphology under kinetics which are mechanically stable in the sense of [9]. The paper is organized as follows.

Chapter 2 is dedicated to preliminaries. Following a synopsis of the notation to be used, Section 2.1 introduces the kinematics and fundamental balance principles which will be needed thereafter. Section 2.2 focuses on the *rate of entropy production* due to the presence of phase boundaries and introduces the associated notion of the *driving traction* acting on a phase boundary. In Section 2.3 a *thermoelastic material* is defined and in Section 2.4 the particular class of thermoelastic materials which will be used in the forthcoming analysis is introduced. Section 2.5 is concerned with the *kinetic relation* and allied *kinetic response function*. In the final section of Chapter 2 the kinematics are specialized to those of antiplane shear and a thermoelastic antiplane shear motion is defined.

Chapter 3 is devoted to the linear stability analysis of an isothermal two-phase process which involves a steadily propagating planar phase boundary in an arbitrary thermoelastic material within the class introduced in Section 3.4. The relevant process, which is a straightforward generalization of that used in the purely mechanical investigation [9], is introduced in Section 3.1, while the class of perturbations to which it will be subjected is put forth in Section 3.2. Each admissible perturbation involves, in general, a disturbance of the configuration

of the phase boundary and of the displacement, velocity and temperature fields in a small neighborhood of the phase boundary. All disturbances are assumed to be small in some appropriate sense. The kinematics of the perturbation are restricted to those of antiplane shear. It is assumed that the post-perturbation process is a thermoelastic antiplane shear and involves only one phase boundary. Sections 3.3 and 3.4 address, respectively, the linearization—about the unperturbed process—of the field equations which hold away from the phase boundary and the jump conditions and kinetic relation which hold on the phase boundary. After a specialization of the base process, a summary of the complete linearized system of field equations, jump conditions, kinetic relation and boundary and initial conditions which describe the process generated by the perturbation is presented in Section 3.5. As in [9], both the inertial and inertia-free cases are included. The combined results of Sections 3.6 and 3.7 show that whenever it is static, regardless of the presence of inertial effects, the base process is linearly stable with respect to all perturbations of the type introduced in Section 3.2. Section 3.8 deals with the case where the base process involves an interface propagating at non-zero velocity. A *normal mode analysis* is performed which leads to a variety of conditions sufficient for the instability of the undisturbed process. These conditions depend on the monotonicity properties of the kinetic response function. Highlighted in Section 3.9 is one set of sufficient conditions which is of particular interest. The relevant conditions alter the conclusions reached in the purely mechanical context considered in [9] in two ways. First, in contrast to the results obtained in the latter setting, instability may arise even when the normal velocity of the phase boundary is a monotonically increasing function of driving traction as long as the temperature dependence in the kinetic response function is of an appropriate nature. Second, the instability that arises in these thermomechanical circumstances occurs only in the *long waves* of the the Fourier decomposition of the moving phase boundary, suggesting that the interface favors a highly wrinkled configuration. This conclusion is akin to that reached in simi-

lar linear stability analyses performed within the context of models for dendritic crystal growth where an otherwise thermally unstable process is stabilized for sufficiently large wave-numbers by the inclusion of surface tension at the interface in lieu of including mechanical effects.[1] The final topic addressed in Section 3.9 pertains to the physical suitability of kinetic response functions which are *mechanically stable* but *thermally unstable*.

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[1] See, for example, LANGER [13], MULLINS & SEKERKA [14] and STRAIN [16].

2. PRELIMINARIES

2.1. Notation, kinematics and balance principles. In the following $I\!R$ and $C\!\!\!\!C$ denote the sets of real and complex numbers. The intervals $(0, \infty)$ and $[0, \infty)$ are represented by $I\!R_+$ and $\bar{\bar{I\!R}}_+$. The symbol $I\!R^n$, with n equal to 2 or 3, represents real n-dimensional space equipped with the standard Euclidean norm. If U is a set, then its closure, interior and boundary are designated by $\bar{U}$, $\overset{\circ}{U}$, and ∂U, respectively. The complement of a set V with respect to U is written as $U \setminus V$. Given a function $\psi : U \to W$ and a subset V of U, $\psi(V)$ stands for the image of V under the map ψ.

Vectors and linear transformations from $I\!R^3$ to $I\!R^3$ (referred to herein as *tensors*) are distinguished from scalars with the aid of boldface type—lower and upper case for vectors and tensors, respectively. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $I\!R^3$, their inner product is then written as $\mathbf{a} \cdot \mathbf{b}$; the Euclidean norm of $\mathbf{a}$ is, further, written as $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The set of unit vectors—that is, vectors with unit Euclidean norm—in $I\!R^3$ is designated by $\mathcal{N}$. The symbol $\mathcal{L}$ refers to the set of tensors, $\mathcal{L}_+$ denotes the set of all tensors with positive determinant, and $\overset{+}{\mathcal{S}}$ stands for the collection of all symmetric positive definite tensors. If $\mathbf{F}$ is in $\mathcal{L}$ then $\mathbf{F}^T$ represents its transpose; if, moreover, $\det \mathbf{F} \neq 0$, then the inverse of $\mathbf{F}$ and its transpose are written as $\mathbf{F}^{-1}$ and $\mathbf{F}^{-T}$, respectively. The notation $\mathbf{a} \otimes \mathbf{b}$ refers to the tensor $\mathbf{A}$, formed by the outer product of $\mathbf{a}$ with $\mathbf{b}$, defined such that $\mathbf{Ac} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for any vector $\mathbf{c}$ in $I\!R^3$. If $\mathbf{A}$ and $\mathbf{B}$ are tensors then their inner product is written as $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr} \mathbf{AB}^T$.

When component notation is used, Greek indices range only over $\{1, 2\}$; summation of repeated indices over the appropriate range is implicit. A subscript preceded by a comma denotes partial differentiation with respect to the corresponding coordinate. Also, a superposed dot signifies partial differentiation with respect to time.

Consider now a body $\mathcal{B}$ which, in a reference configuration, occupies a region $\mathcal{R}$ contained in $I\!R^3$. A motion of $\mathcal{B}$ on a time interval $\mathcal{T} \subset I\!R$ is characterized by

a one-parameter family of invertible mappings $\hat{\mathbf{y}}(\cdot, t) : \mathcal{R} \to \mathcal{R}_t$, with

$$\hat{\mathbf{y}}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \quad \forall (\mathbf{x}, t) \in \mathcal{M}, \tag{2.1.1}$$

where $\mathcal{M} = \mathcal{R} \times \mathcal{T}$ represents the *trajectory* of the motion. Assume that the *deformation* $\hat{\mathbf{y}}$, or equivalently the *displacement* $\mathbf{u}$, is continuous and possesses piecewise continuous first and second partial derivatives on $\mathcal{M}$. Let S_t be the set of points contained in $\mathcal{R}$ defined so that, at each instant t in $\mathcal{T}$, $\hat{\mathbf{y}}(\cdot, t)$ is twice continuously differentiable on the set $\mathcal{R} \setminus S_t$. Let the set Σ be defined by

$$\Sigma = \{(\mathbf{x}, t) \mid \mathbf{x} \in S_t, t \in \mathcal{T}\}. \tag{2.1.2}$$

Introduce the *deformation gradient tensor* $\mathbf{F} : \mathcal{M} \setminus \Sigma \to \mathcal{L}$ by

$$\mathbf{F}(\mathbf{x}, t) = \nabla \hat{\mathbf{y}}(\mathbf{x}, t) \quad \forall (\mathbf{x}, t) \in \mathcal{M} \setminus \Sigma, \tag{2.1.3}$$

where the associated *Jacobian determinant*, $J : \mathcal{M} \setminus \Sigma \to I\!\!R$, of $\hat{\mathbf{y}}$ is restricted to be strictly positive on its domain of definition:

$$J(\mathbf{x}, t) = \det \mathbf{F}(\mathbf{x}, t) > 0 \quad \forall (\mathbf{x}, t) \in \mathcal{M} \setminus \Sigma.$$

Hence, $\mathbf{F} : \mathcal{M} \setminus \Sigma \to \mathcal{L}_+$. The *left Cauchy-Green tensor* $\mathbf{G} : \mathcal{M} \setminus \Sigma \to \overset{+}{\mathcal{S}}$ corresponding to the deformation $\hat{\mathbf{y}}$ is given by

$$\mathbf{G}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t) \mathbf{F}^T(\mathbf{x}, t) \quad \forall (\mathbf{x}, t) \in \mathcal{M} \setminus \Sigma. \tag{2.1.4}$$

The *deformation invariants* associated with $\hat{\mathbf{y}}$ exist on $\mathcal{M} \setminus \Sigma$ and are supplied through the fundamental scalar invariants of $\mathbf{G}$:

$$I_1(\mathbf{G}) = \operatorname{tr} \mathbf{G}, \quad I_2(\mathbf{G}) = \tfrac{1}{2} \left((\operatorname{tr} \mathbf{G})^2 - \operatorname{tr}(\mathbf{G}^2) \right), \quad I_3(\mathbf{G}) = \det \mathbf{G}. \tag{2.1.5}$$

With the above kinematic antecedents in place introduce the *nominal mass density* $\rho : \mathcal{R} \rightarrow I\!\!R_+$, the *nominal body force per unit mass* $\mathbf{b} : \mathcal{M} \rightarrow I\!\!R^3$, and the *nominal stress tensor* $\mathbf{S} : \mathcal{M} \setminus \Sigma \rightarrow \mathcal{L}$, and suppose that ρ is constant on $\mathcal{R}$ and $\mathbf{b}$ is continuous on $\mathcal{M}$, while $\mathbf{S}$ is piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and has a piecewise continuous gradient on $\mathcal{M}$. Let ρ_* be the mass density in the deformed configuration associated with $\hat{\mathbf{y}}$. Given a *regular* subregion $\mathcal{P}$ of $\mathcal{R}$, with $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$, let $\mathbf{m} : \partial\mathcal{P} \rightarrow \mathcal{N}$ denote the unit outward normal to $\partial\mathcal{P}$. Then the global balance laws of mass, linear momentum, and angular momentum require that

$$\int_{\mathcal{P}} \rho \, dV = \int_{\hat{\mathbf{y}}(\mathcal{P})} \rho_* \, dV \quad \text{on} \quad \mathcal{T}, \tag{2.1.6}$$

$$\int_{\partial\mathcal{P}} \mathbf{Sm} \, dA + \int_{\mathcal{P}} \rho\mathbf{b} \, dV = \overline{\int_{\mathcal{P}} \rho\dot{\mathbf{u}} \, dV} \quad \text{on} \quad \mathcal{T}, \tag{2.1.7}$$

and

$$\int_{\partial\mathcal{P}} \hat{\mathbf{y}} \wedge \mathbf{Sm} \, dA + \int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\mathbf{b} \, dV = \overline{\int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\dot{\mathbf{u}} \, dV} \quad \text{on} \quad \mathcal{T}, \tag{2.1.8}$$

respectively, for every such regular subregion $\mathcal{P}$ contained in $\mathcal{R}$.

Next, introduce the *nominal internal energy per unit mass* $\varepsilon : \mathcal{M} \setminus \Sigma \rightarrow I\!\!R$, the *nominal heat flux* $\mathbf{q} : \mathcal{M} \setminus \Sigma \rightarrow I\!\!R^3$, and the *nominal heat supply per unit mass* $r : \mathcal{M} \setminus \Sigma \rightarrow I\!\!R$. Suppose that ε and $\mathbf{q}$ are piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and have piecewise continuous partial derivatives on $\mathcal{M}$, and that r is continuous on $\mathcal{M}$. The first law of thermodynamics requires that

$$\int_{\partial\mathcal{P}} (\mathbf{Sm} \cdot \dot{\mathbf{u}} + \mathbf{q} \cdot \mathbf{m}) \, dA + \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \dot{\mathbf{u}} + r) \, dV = \overline{\int_{\mathcal{P}} \rho(\varepsilon + \tfrac{1}{2}|\dot{\mathbf{u}}|^2) \, dV} \quad \text{on} \quad \mathcal{T}, \tag{2.1.9}$$

for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$ such that $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$.

Finally, introduce the *nominal entropy per unit mass* $\eta : \mathcal{M} \setminus \Sigma \to I\!\!R$ and the *nominal absolute temperature* $\theta : \mathcal{M} \to I\!\!R$. Stipulate that η is piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and has piecewise continuous first partial derivatives on $\mathcal{M}$, and that θ is continuous on $\mathcal{M}$ with piecewise continuous first partial derivatives on $\mathcal{M}$. The *Clausius-Duhem* version of the second law of thermodynamics requires that the *rate of entropy production* $\Gamma(\cdot; \mathcal{P}) : \mathcal{T} \to I\!\!R$ satisfies

$$\Gamma(\cdot; \mathcal{P}) = \overline{\int_{\mathcal{P}} \rho \dot{\eta} \, dV} - \int_{\partial \mathcal{P}} \frac{\mathbf{q} \cdot \mathbf{m}}{\theta} \, dA - \int_{\mathcal{P}} \frac{\rho r}{\theta} \, dV \geq 0 \quad \text{on} \quad \mathcal{T}, \qquad (2.1.10)$$

for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$ such that $\partial \mathcal{P} \cap S_t$ a set of measure zero in $\partial \mathcal{P}$ for each t in $\mathcal{T}$.

Localization of the balance laws (2.1.6)–(2.1.9) and the imbalance law (2.1.10) at an arbitrary point contained in the interior of $\mathcal{M} \setminus \Sigma$ yields the following familiar field equations and field inequality:

$$\rho = \rho_*(\hat{\mathbf{y}}) J \quad \text{on} \quad \mathcal{M} \setminus \Sigma,$$

$$\nabla \cdot \mathbf{S} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{on} \quad \mathcal{M} \setminus \Sigma,$$

$$\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \qquad (2.1.11)$$

$$\mathbf{S} \cdot \dot{\mathbf{F}} + \nabla \cdot \mathbf{q} + \rho r = \rho \dot{\varepsilon} \quad \text{on} \quad \mathcal{M} \setminus \Sigma,$$

$$\nabla \cdot \left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta} \leq \rho \dot{\eta} \quad \text{on} \quad \mathcal{M} \setminus \Sigma.$$

Suppose, from now on, that the set S_t is a regular surface for every t in $\mathcal{T}$. The set Σ then represents the trajectory of a surface of discontinuity in $\mathbf{F}$, $\mathbf{S}$ and, perhaps, ε, $\mathbf{q}$ and η. Let $g(\cdot, t)$ denote a generic field quantity $g(\cdot, t) : S_t \to I\!\!R$ which is discontinuous across S_t at the instant t in $\mathcal{T}$. Define the *jump* $[\![g(\cdot, t)]\!]$ of $g(\cdot, t)$ across S_t by

$$[\![g(\mathbf{x}, t)]\!] = \lim_{h \searrow 0} \left(g(\mathbf{x} + h\mathbf{n}(\mathbf{x}, t), t) - g(\mathbf{x} - h\mathbf{n}(\mathbf{x}, t), t) \right) \quad \forall (\mathbf{x}, t) \in \Sigma, \quad (2.1.12)$$

where $n(\cdot, t) : S_t \to \mathcal{N}$ is a unit normal to S_t at each t in $\mathcal{T}$. Then, localization of (2.1.6)–(2.1.10) at an arbitrary element of Σ yields the following jump conditions

$$[\rho_*(\hat{y})J] = 0 \quad \text{on} \quad \Sigma,$$

$$[\mathbf{Sn}] + \rho V_n [\dot{\mathbf{u}}] = 0 \quad \text{on} \quad \Sigma,$$

$$[\mathbf{Sn} \cdot \dot{\mathbf{u}}] + \rho V_n [\varepsilon + \tfrac{1}{2}|\dot{\mathbf{u}}|^2] + [\mathbf{q} \cdot \mathbf{n}] = 0 \quad \text{on} \quad \Sigma, \qquad (2.1.13)$$

$$\rho V_n [\eta] + \frac{1}{\theta}[\mathbf{q} \cdot \mathbf{n}] \leq 0 \quad \text{on} \quad \Sigma,$$

where $V_n(\cdot, t) : S_t \to I\!R$ is the component of the velocity of the surface S_t in the direction of $n(\cdot, t)$ at the instant t in $\mathcal{T}$.

Equations $(2.1.11)_1$ and $(2.1.13)_1$ are, evidently, completely decoupled from equations $(2.1.11)_{2,3,4,5}$ and $(2.1.13)_{2,3,4}$; that is, given a solution to, say, a boundary value problem involving $(2.1.11)_{2,3,4,5}$ and $(2.1.13)_{2,3,4}$, ρ_* can be calculated *a posteriori*. For this reason equations $(2.1.11)_1$ and $(2.1.13)_1$ will be disregarded in the subsequent analysis.

In this investigation an *inertia-free* motion is defined as one wherein the inertial terms on the right hand sides of the global balance equations (2.1.7) and (2.1.8) are replaced by the zero vector. In the context of an inertia-free motion the field equation $(2.1.11)_2$ simplifies to read

$$\nabla \cdot \mathbf{S} + \rho \mathbf{b} = 0 \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \qquad (2.1.14)$$

and the jump condition $(2.1.13)_2$ becomes

$$[\mathbf{Sn}] = 0 \quad \text{on} \quad \Sigma. \qquad (2.1.15)$$

Equations $(2.1.11)_{1,3,4,5}$ and $(2.1.13)_{1,3,4}$ remain, of course, unaltered.

In addition to the jump conditions given in (2.1.13) in the inertial case or $(2.1.13)_{1,3,4}$ and (2.1.15) in the inertia-free case, the stipulated continuity of $\hat{y}$

and θ gives the following *kinematic* jump conditions

$$[\mathbf{u}] = 0 \quad \text{on} \quad \Sigma, \quad [\theta] = 0 \quad \text{on} \quad \Sigma. \qquad (2.1.16)$$

2.2. Rate of entropy production and driving traction. Using the field equations (2.1.11), the jump conditions (2.1.13) , and the assumed smoothness of the deformation $\hat{\mathbf{y}}$, ABEYARATNE & KNOWLES [1] have demonstrated that for any continuum the rate of entropy production $\Gamma(\cdot; \mathcal{P})$ can, for any regular region $\mathcal{P}$ contained in $\mathcal{R}$, be represented in the form

$$\Gamma(t; \mathcal{P}) = \Gamma_{\text{loc}}(t; \mathcal{P}) + \Gamma_{\text{con}}(t; \mathcal{P}) + \Gamma_s(t; \mathcal{P}) \quad \forall t \in \mathcal{T}, \qquad (2.2.1)$$

where $\Gamma_{\text{loc}}(\cdot; \mathcal{P})$, $\Gamma_{\text{con}}(\cdot; \mathcal{P})$, and $\Gamma_s(\cdot; \mathcal{P})$ are defined by

$$\Gamma_{\text{loc}}(\cdot; \mathcal{P}) = \int_{\mathcal{P} \backslash S_t} \frac{1}{\theta}(\mathbf{S} \cdot \dot{\mathbf{F}} - \rho(\dot{\psi} + \dot{\theta}\eta)) \, dV \quad \text{on} \quad \mathcal{T},$$

$$\Gamma_{\text{con}}(\cdot; \mathcal{P}) = \int_{\mathcal{P} \backslash S_t} \frac{1}{\theta^2} \mathbf{q} \cdot \nabla\theta \, dV \quad \text{on} \quad \mathcal{T}, \qquad (2.2.2)$$

$$\Gamma_s(\cdot; \mathcal{P}) = \int_{\mathcal{P} \cap S_t} \frac{1}{\theta}(\rho[\psi] - \langle\!\langle \mathbf{S} \rangle\!\rangle \cdot [\mathbf{F}]) V_n \, dV \quad \text{on} \quad \mathcal{T},$$

with $\psi : \mathcal{M} \backslash \Sigma \to I\!\!R$ representing the *nominal Helmholtz free energy per unit mass* in defined in terms of ϵ, θ and η by

$$\psi = \epsilon - \theta\eta \quad \text{on} \quad \mathcal{M} \backslash \Sigma, \qquad (2.2.3)$$

and, where—given a generic field quantity $g(\cdot, t) : S_t \to I\!\!R$ which jumps across S_t at the instant t in $\mathcal{T}$—$\langle\!\langle g(\cdot, t) \rangle\!\rangle$ is defined through

$$\langle\!\langle g(\mathbf{x}, t) \rangle\!\rangle = \lim_{h \searrow 0} \tfrac{1}{2}\left(g(\mathbf{x} + h\mathbf{n}(\mathbf{x}, t), t) + g(\mathbf{x} - h\mathbf{n}(\mathbf{x}, t), t)\right) \quad \forall(\mathbf{x}, t) \in \Sigma. \quad (2.2.4)$$

The representation (2.2.1) additively decomposes the total rate of entropy production $\Gamma(\cdot; \mathcal{P})$, at the instant t in $\mathcal{T}$, in the regular region $\mathcal{P}$ contained in $\mathcal{R}$ into three parts. The first two terms in the decomposition $\Gamma_{\text{loc}}(\cdot; \mathcal{P})$ and $\Gamma_{\text{con}}(\cdot; \mathcal{P})$ are the contributions to the rate of entropy production due, respectively, to local mechanical dissipation and heat conduction away from the surface S_t, while the third term $\Gamma_s(\cdot; \mathcal{P})$ represents the entropy production rate due to the motion of the surface S_t.

Motivated by (2.2.2)$_3$ define the *driving traction* $f(\cdot, t) : S_t \to \mathbb{R}$ which acts on the surface S_t at the instant t in $\mathcal{T}$ by

$$f(\cdot, t) = \rho[\![\psi(\cdot, t)]\!] - \langle\!\langle \mathbf{S}(\cdot, t) \rangle\!\rangle \cdot [\![\mathbf{F}(\cdot, t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}. \tag{2.2.5}$$

In the absence of inertial effects it can be demonstrated that (2.2.5) reduces to

$$f(\cdot, t) = \rho[\![\psi(\cdot, t)]\!] - \overset{\pm}{\mathbf{S}}(\cdot, t) \cdot [\![\mathbf{F}(\cdot, t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}, \tag{2.2.6}$$

where $\overset{+}{\mathbf{S}}(\cdot, t)$ (*resp.*, $\overset{-}{\mathbf{S}}(\cdot, t)$) is the limiting value of the field $\mathbf{S}(\cdot, t)$ on the side of the interface into which the unit normal $\mathbf{n}(\cdot, t)$ is (*resp.*, is not) directed at the instant t in $\mathcal{T}$.

Now, from (2.2.1) and (2.2.2)$_3$, localization of the imbalance law (2.1.10) at an arbitrary element of Σ yields the following alternative to (2.1.13)$_4$:

$$f V_n \geq 0 \quad \text{on} \quad \Sigma, \tag{2.2.7}$$

with f given by (2.2.5) or (2.2.6) depending on whether inertial effects are included or not. Observe , from (2.2.2), that under isothermal conditions the total rate of entropy production $\Gamma(\cdot; \mathcal{P})$ in a region $\mathcal{P}$ takes the form of the rate of mechanical dissipation per unit temperature.

2.3. Finite thermoelasticity. Let $\mathcal{B}$ be composed of a homgeneous *thermoelastic* material. Then there exists a *Helmholtz free energy potential*

$\hat{\psi} : \mathcal{L}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that the nominal Helmholtz free energy per unit mass ψ, the nominal stress tensor $\mathbf{S}$, and the nominal entropy per unit mass are given in terms of $\hat{\psi}$ as follows:

$$\psi = \hat{\psi}(\mathbf{F}, \theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma,$$

$$\mathbf{S} = \rho\hat{\psi}_{\mathbf{F}}(\mathbf{F}, \theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \tag{2.3.1}$$

$$\eta = -\hat{\psi}_\theta(\mathbf{F}, \theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma.$$

It is assumed that $\hat{\psi}$ is once continuously differentiable and piecewise twice continuously differentiable on $\mathcal{L}_+ \times \mathbb{R}_+$. The nominal heat flux $\mathbf{q}$ is, for a thermoelastic material, given by a *heat flux response function* $\hat{\mathbf{q}} : \mathcal{L}_+ \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ so that

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F}, \theta, \nabla\theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \tag{2.3.2}$$

It is assumed that $\hat{\mathbf{q}}$ is piecewise twice continuously differentiable on its domain of definition.

Observe that a thermoelastic material is defined in a manner such that the rate of entropy production $\Gamma_{\text{loc}}(\cdot; \mathcal{P})$ in a region $\mathcal{P}$ due to mechanical dissipation is identically zero on $\mathcal{T}$. Hence, the localization of the imbalance law (2.1.10) at a point contained in the interior of $\mathcal{M} \setminus \Sigma$ yields, with the aid of (2.3.2), the inequality

$$\hat{\mathbf{q}}(\cdot, \cdot, \mathbf{d}) \cdot \mathbf{d} \geq 0 \quad \text{on} \quad \mathcal{L}_+ \times \mathbb{R}_+ \quad \forall \mathbf{d} \in \mathbb{R}^3 \tag{2.3.3}$$

as a condition necessary for the satisfaction of the second law of thermodynamics. The response function $\hat{\mathbf{q}}$ is assumed to be specified so that (2.3.3) holds; then, inequality $(2.1.11)_5$ is automatically satisfied and can be ignored in the following.

For remarks regarding the consequences of *objectivity* on the properties of the potential $\hat{\psi}$ and response function $\hat{\mathbf{q}}$, see JIANG [11].

2.4. Constitutive specialization. To facilitate the ensuing analysis suppose, henceforth, that the homogeneous thermoelastic body $\mathcal{B}$ is thermomechanically isotropic. Then the Helmholtz free energy potential $\hat{\psi}$ and heat flux response

function $\hat{\mathbf{q}}$ can depend on the deformation gradient $\mathbf{F}$ only through the deformation invariants $I_k(\mathbf{G})$ defined in (2.1.5). Assume henceforth that both $\hat{\psi}$ and $\hat{\mathbf{q}}$ are independent of the second deformation invariant $I_2(\mathbf{G})$. Suppose, moreover, that the Helmholtz free energy potential $\hat{\psi}$ can be represented in terms of three functions $\tilde{\psi} : I\!R_+ \times I\!R_+ \to I\!R$, $\tilde{g} : I\!R_+ \times I\!R_+ \to I\!R$ and $\tilde{\tilde{g}} : I\!R_+ \times I\!R_+ \to I\!R$ in the form

$$\hat{\psi}(\mathbf{F},\theta) = \tilde{\psi}(I_1(\mathbf{G}),\theta) + \tilde{\psi}_{I_1}(I_1(\mathbf{G}),\theta)\tilde{g}(I_3(\mathbf{G}),\theta) + \tilde{\tilde{g}}(I_3(\mathbf{G}),\theta)$$

$$\forall(\mathbf{F},\theta) \in \mathcal{L}_+ \times I\!R_+, \qquad (2.4.1)$$

and that the heat flux response function $\hat{\mathbf{q}}$ can be expressed in terms of a function $\tilde{\phi} : I\!R_+ \times I\!R_+ \times I\!R_+ \to I\!R$ via

$$\hat{\mathbf{q}}(\mathbf{F},\theta,\mathbf{d}) = \tilde{\phi}(I_1(\mathbf{G}), I_3(\mathbf{G}),\theta)\mathbf{d} \quad \forall(\mathbf{F},\theta,\mathbf{d}) \in \mathcal{L}_+ \times I\!R_+ \times I\!R^3. \qquad (2.4.2)$$

In (2.4.1) and (2.4.2) $\mathbf{G}$ is the left Cauchy-Green tensor defined in terms of the deformation gradient tensor $\mathbf{F}$ by (2.1.4). In accordance with the stipulated smoothness of $\hat{\psi}$ and $\hat{\mathbf{q}}$, the functions $\tilde{\psi}$, $\tilde{g}$ and $\tilde{\tilde{g}}$ are taken to be once continuously differentiable and piecewise twice continuously differentiable on $I\!R_+ \times I\!R_+$, while $\tilde{\phi}$ is taken to be continuous and piecewise twice continuously differentiable on $I\!R_+ \times I\!R_+ \times I\!R_+$. Assume, in addition, that the functions $\tilde{\psi}$, $\tilde{g}$ and $\tilde{\tilde{g}}$ comply, for each θ in $I\!R_+$, with the following *isochoric* restrictions:

$$\tilde{\psi}(3,\theta) = 0, \quad \tilde{g}(1,\theta) = 0, \quad \tilde{g}_{I_3}(1,\theta) = -1, \quad \tilde{\tilde{g}}(1,\theta) = \tilde{\tilde{g}}_{I_3}(1,\theta) = 0. \qquad (2.4.3)$$

In what follows, attention will be restricted to homogeneous isotropic thermoelastic materials wherein the Helmholtz free energy potential $\hat{\psi}$ obeys (2.4.3). A particular material of this type was studied by JIANG [11].

The nominal stress response of $\mathcal{B}$ is then determined, with the aid of (2.3.1)$_2$ and (2.4.1), by

$$\mathbf{S} = 2\rho\left(\chi_1(I_1(\mathbf{G}), I_3(\mathbf{G}),\theta)\mathbf{F} + \chi_2(I_1(\mathbf{G}), I_3(\mathbf{G}),\theta)\mathbf{F}^{-T}\right) \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \quad (2.4.4)$$

where the functions $\chi_\alpha : I\!\!R_+ \times I\!\!R_+ \times I\!\!R_+ \to I\!\!R$ are defined as follows:

$$\chi_1(I_1(\mathbf{G}), I_3(\mathbf{G}), \theta) = \tilde{\psi}_{I_1}(I_1, \theta) + \tilde{\psi}_{I_1 I_1}(I_1, \theta)\tilde{g}(I_3, \theta)$$

$$\forall (I_1, I_3, \theta) \in I\!\!R_+ \times I\!\!R_+ \times I\!\!R_+,$$

$$\chi_2(I_1(\mathbf{G}), I_3(\mathbf{G}), \theta) = \tilde{\psi}_{I_1}(I_1, \theta)\tilde{g}_{I_3}(I_3, \theta) + \tilde{\tilde{g}}_{I_3}(I_3, \theta)$$

$$\forall (I_1, I_3, \theta) \in I\!\!R_+ \times I\!\!R_+ \times I\!\!R_+.$$

<div align="right">(2.4.5)</div>

Observe that, for an isochoric deformation—where $I_3 = 1$ on $\mathcal{M} \setminus \Sigma$, use of (2.4.3) in (2.4.5) reduces (2.4.4) to read

$$\mathbf{S} = 2\rho\tilde{\psi}_{I_1}(I_1, \theta)\left(\mathbf{F} - \mathbf{F}^{-T}\right) \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \tag{2.4.6}$$

Following the work of JIANG & KNOWLES [12] in the purely mechanical setting, it can be readily shown that a special thermoelastic material of the type characterized by (2.4.1)–(2.4.3) satisfies the *Baker-Ericksen inequalities* at all absolute temperatures if and only if

$$\tilde{\psi}_{I_1}(I_1, \theta) + \tilde{\psi}_{I_1 I_1}(I_1, \theta)\tilde{g}(I_3, \theta) > 0 \quad \forall (I_1, I_3, \theta) \in \mathcal{U} \times I\!\!R_+, \tag{2.4.7}$$

where the set $\mathcal{U}$ is given by

$$\mathcal{U} = \{(I_1, I_3) | \, 0 < I_3 < (I_1/3)^3\}.$$

Choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and consider the response of the thermoelastic material at hand to a simple shear deformation $\hat{\mathbf{y}}$ given as follows

$$\hat{\mathbf{y}}(\mathbf{x}, t) = (1 + \gamma \mathbf{e}_3 \otimes \mathbf{e}_1)\mathbf{x} \quad \forall (\mathbf{x}, t) \in \mathcal{M}, \tag{2.4.8}$$

where the constant γ—assumed non-negative without loss of generality—denotes the amount of shear. Note that the foregoing deformation is isochoric. From

$(2.3.1)_2$, (2.4.1), (2.4.3) and (2.4.6) the nominal shear stress corresponding to the deformation $\hat{y}$ is, therefore, for each γ in $\mathbb{R}_+$, found to be

$$\mathbf{e}_3 \cdot \mathbf{S}\mathbf{e}_1 = 2\rho\gamma\tilde{\psi}_{I_1}(3 + \gamma^2, \theta) =: \tau(\gamma, \theta). \tag{2.4.9}$$

where θ takes on some positive value. The function $\tau : \bar{\mathbb{R}}_+ \times \mathbb{R}_+ \to \mathbb{R}$ will be referred to as the *shear stress response function* of the special thermoelastic material at hand in simple shear. An immediate consequence of $(2.4.3)_1$ and (2.4.9) is that $\rho\tilde{\psi}$ can be expressed via

$$\rho\tilde{\psi}(I_1, \theta) = \int_0^{\sqrt{I_1 - 3}} \tau(\kappa, \theta)\, d\kappa \quad \forall (I_1, \theta) \in [3, \infty) \times \mathbb{R}_+, \tag{2.4.10}$$

so that the nominal stress response of such a material, in *all* three dimensional deformations and absolute temperatures, is completely characterized by specifying a shear stress response function τ along with the functions $\tilde{g}$ and $\tilde{\tilde{g}}$ introduced in (2.4.1). Now, define the *secant modulus in shear* $M : \bar{\mathbb{R}}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$M(\gamma, \theta) = 2\rho\tilde{\psi}_{I_1}(3 + \gamma^2, \theta) \quad \forall (\gamma, \theta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}_+. \tag{2.4.11}$$

Observe that, in compliance with the stipulated smoothness of $\tilde{\psi}$, both τ and M must be continuous and piecewise continuously differentiable on $\bar{\mathbb{R}}_+ \times \mathbb{R}_+$. From (2.2.9) and (2.2.11) that the shear stress response function τ must also satisfy

$$\tau(0, \theta) = 0 \quad \forall \theta \in \mathbb{R}_+, \quad \tau_\gamma(0, \theta) = M(0, \theta) \quad \forall \theta \in \mathbb{R}_+. \tag{2.4.12}$$

Note, also, that for the simple shear deformation defined via (2.1.1) and (2.4.5), the *Baker-Ericksen inequality* (2.4.10) reduces, with the aid of (2.4.11) and $(2.4.3)_4$, to a relation which involves only M: viz.,

$$M(\gamma, \theta) > 0 \quad \forall (\gamma, \theta) \in \mathbb{R}_+^2. \tag{2.4.13}$$

Restrict attention in the sequel to these special materials for which the infinites-imal shear modulus is positive; *i.e.*, require that

$$M(0, \theta) > 0 \quad \forall \theta \in I\!\!R_+. \qquad (2.4.14)$$

Despite the significant restrictions which have been placed upon the class of materials which will be considered in this investigation, no presuppositions have been made regarding the sign of the derivative with respect to its first argument—where it exists—of the shear stress response function corresponding to the thermoelastic material defined in compliance with (2.4.1)–(2.4.3) and (2.4.10). JIANG [11] has shown that the monotonicity of $\tau(\cdot, \theta)$ is, for fixed θ in $I\!\!R_+$, related directly to the ellipticity of the material which it characterizes. If, in particular, $\tau(\cdot, \theta)$ is not a monotonically increasing function on its domain of definition—for some range of θ—then the associated material is non-elliptic. With this in mind, let (θ_m, θ_M) be contained in $I\!\!R_+$ and define functions $\underset{*}{\gamma} : (\theta_m, \theta_M) \to I\!\!R_+$ and $\overset{*}{\gamma} : (\theta_m, \theta_M) \to I\!\!R_+$ such that

$$\underset{*}{\gamma}(\theta) < \overset{*}{\gamma}(\theta) \quad \forall \theta \in (\theta_m, \theta_M). \qquad (2.4.15)$$

Next, define three plane open subsets A_1, A_2, and A_3 of the *shear strain-temperature* quadrant as follows:

$$A_1 = \{(\gamma, \theta) \mid 0 < \gamma < \underset{*}{\gamma}(\theta), \theta \in (\theta_m, \theta_M)\},$$

$$A_2 = \{(\gamma, \theta) \mid \underset{*}{\gamma}(\theta) < \gamma < \overset{*}{\gamma}(\theta), \theta \in (\theta_m, \theta_M)\}, \qquad (2.4.16)$$

$$A_3 = \{(\gamma, \theta) \mid \overset{*}{\gamma}(\theta) < \gamma < \infty, \theta \in (\theta_m, \theta_M)\}.$$

This investigation will make use of a particular subclass of non-elliptic thermoe-lastic materialsof the above special form wherein the relevant shear stress response function τ is taken to be continuous on $\bar{I\!\!R}_+ \times I\!\!R_+$ and continuously differentiable on $A_1 \cup A_2 \cup A_3$ and is required to obey the following monotonicity requirements

$$\tau_\gamma > 0 \quad \text{on} \quad A_1 \cup A_3,$$
$$\tau_\gamma < 0 \quad \text{on} \quad A_2. \qquad (2.4.17)$$

Assume, also, that $\tau(\cdot, \theta)$ is monotonically increasing on $I\!\!R_+$ for all θ in $I\!\!R_+ \setminus [\theta_m, \theta_M]$. Let the *nominal conductivity in shear* $\tilde{k} : I\!\!R \times I\!\!R_+ \to I\!\!R_+$ and the *nominal specific heat per unit mass in shear* $\tilde{c} : I\!\!R \times I\!\!R_+ \to I\!\!R_+$ of the thermoelastic material at hand be defined as follows

$$\tilde{k}(\gamma, \theta) = \tilde{\phi}(3 + \gamma^2, 1, \theta) \quad \forall (\gamma, \theta) \in I\!\!R \times I\!\!R_+,$$

$$\tilde{c}(\gamma, \theta) = -\theta \tilde{\psi}_{\theta\theta}(3 + \gamma^2, \theta) \quad \forall (\gamma, \theta) \in I\!\!R \times I\!\!R_+. \tag{2.4.18}$$

Suppose that $\tilde{k}$ and $\tilde{c}$ are both continuous on $A_1 \cup A_2 \cup A_3$ and piecewise continuous on $I\!\!R_+ \times I\!\!R_+$. In compliance with (2.3.3) let $\tilde{k}$ be positive on its domain of definition. Suppose, in addition, that $\tilde{c}$ is positive on its domain of definition.

The sets A_1 and A_3 are referred to as the *high* and *low strain phases* of the thermoelastic material specified by (2.4.1)–(2.4.3) and (2.4.10). These, together with the set of shear strain-temperature pairs in $(I\!\!R_+ \times I\!\!R_+) \setminus (\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3)$ comprise the *elliptic phases* of such a material. A thermoelastic material of the type at hand which is defined so that τ, $\tilde{k}$ and $\tilde{c}$ have the properties set forth above will be referred to herein as a *three-phase* thermoelastic material. See Figure 1 for a graph of $\tau(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M) typical of those which specify three-phase thermoelastic materials. Consult Figure 2 for a division of the shear strain-temperature quadrant into regions of monotonicity of $\tau(\cdot, \theta)$ for fixed θ.

2.5. Completion of constitutive assumptions via the kinetic relation. Let $\mathcal{B}$ be composed of a three-phase thermoelastic material and consider a motion of $\mathcal{B}$ which involves a moving surface of discontinuity S_t in one or all of the field quantities $\mathbf{F}(\cdot, t)$, $\dot{\mathbf{u}}(\cdot, t)$, $\mathbf{S}(\cdot, t)$, $\psi(\cdot, t)$, $\eta(\cdot, t)$, and $\mathbf{q}(\cdot, t)$ at each instant t in $\mathcal{T}$. Assume that S_t separates high and low strain elliptic phases in the material at hand. In the context of such a motion it is necessary (see [1–4]) to supplement, in some fashion, the constitutive information provided in Section 2.4. An approach to this taken by ABEYARATNE & KNOWLES [1] entails the provision of a *kinetic relation*. Two basic cases motivated by [1] can be considered: in the

first a constitutive response function $\tilde{V} : I\!\!R \times (\theta_m, \theta_M) \to I\!\!R$ is specified so that

$$V_n = \tilde{V}(\tfrac{f}{\theta}, \theta) \quad \forall(\tfrac{f}{\theta}, \theta) \in I\!\!R \times (\theta_m, \theta_M), \qquad (2.5.1)$$

while, in the second a constitutive response function $\tilde{\varphi} : I\!\!R \times (\theta_m, \theta_M) \to I\!\!R$ is furnished so that

$$f = \theta\tilde{\varphi}(V_n, \theta) \quad \forall(V_n, \theta) \in I\!\!R \times (\theta_m, \theta_M). \qquad (2.5.2)$$

The functions $\tilde{V}$ and $\tilde{\varphi}$ are referred to as *kinetic response functions*. Since the three-phase thermoelastic material can lose ellipticity only for absolute temperatures θ in (θ_m, θ_M), the kinetic response functions $\tilde{V}$ and $\tilde{\varphi}$ are defined only on $I\!\!R \times (\theta_m, \theta_M)$. Both varieties of kinetic response functions will be considered in this investigation. If $\tilde{V}$ is such that $\tilde{V}(\Phi, \theta)\Phi \geq 0$ for all (Φ, θ) in $I\!\!R \times (\theta_m, \theta_M)$ then (2.2.6) is automatically satisfied and $\tilde{V}$ is referred to as *admissible*. If $\tilde{\varphi}(V, \theta)V \geq 0$ for all (V, θ) in $I\!\!R \times (\theta_m, \theta_M)$, $\tilde{\varphi}$ is, similarly, referred to as admissible. If an admissible kinetic response function $\tilde{V}$ (or $\tilde{\varphi}$) is continuous on $I\!\!R \times (\theta_m, \theta_M)$, then it must satisfy $\tilde{V}(0, \theta) = 0$ (or $\tilde{\varphi}(0, \theta) = 0$) for all θ in (θ_m, θ_M). If, in addition, to being admissible, $\tilde{V}$ (or $\tilde{\varphi}$) is continuously differentiable on $I\!\!R \times (\theta_m, \theta_M)$, then $\tilde{V}_\Phi(0, \theta) \geq 0$ and $\tilde{V}_\Theta(0, \theta) = 0$ (or $\tilde{\varphi}_V(0, \theta) \geq 0$ and $\tilde{\varphi}_\theta(0, \theta) = 0$) for all θ in (θ_m, θ_M)—here $\tilde{V}_\Phi$ and $\tilde{\varphi}_V$ refer to the first partial derivatives of $\tilde{V}$ and $\tilde{\varphi}$ with respect to their first arguments while $\tilde{V}_\Theta$ and $\tilde{\varphi}_\theta$ refer to the first partial derivatives of $\tilde{V}$ and $\tilde{\varphi}$ with respect to their second arguments. Otherwise, admissiblility implies nothing with regard to the sign of the derivative of a smooth kinetic response function. All kinetic response functions considered herein are assumed to be admissible. See Figure 3 and Figure 4 for illustrative graphs of $\tilde{V}(\cdot, \theta_0)$ and $\tilde{\varphi}(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M).

2.6. Thermoelastic antiplane shear motions of a special thermoelastic material. Suppose, from now on, that $\mathcal{R}$ is a cylindrical region and choose a rectangular Cartesian frame $X = \{0; e_1, e_2, e_3\}$ so that the unit base vector e_3

is parallel to the generatrix of $\mathcal{R}$. A dynamical process will be referred to as a *thermoelastic antiplane shear* normal to the plane spanned by the base vectors e_1 and e_2 if the deformation $\hat{y}$ is of the form

$$\hat{y}(\mathbf{x}, t) = \mathbf{x} + u(x_1, x_2, t)e_3 \quad \forall(\mathbf{x}, t) \in \mathcal{M}, \tag{2.6.1}$$

and the nominal Helmholtz free energy per unit mass ψ, nominal entropy per unit mass η, nominal absolute temperature θ, and nominal heat flux vector $\mathbf{q}$ are—like the displacement field associated with (2.6.1)—independent of the x_3–coordinate. The non-trivial component of displacement u in (2.6.1) will be referred to as the *out-of-plane* displacement field. Inspection of (2.6.1) reveals that any discontinuities in the gradient and, perhaps, time derivative of $\hat{y}$ must result from discontinuities in the spatial derivatives out-of-plane displacement field and, hence, occur across surfaces which do not vary with the x_3–coordinate; similarly, because of their independence of the x_3–coordinate, any discontinuities in ψ, η or $\mathbf{q}$ must occur across surfaces which do not vary with the x_3–coordinate. Let S_t denote a surface across which at least one of the above field quantities jumps at the instant t in $\mathcal{T}$ and let Σ be defined as in (2.1.2).

Following the work of JIANG [11] in the inertia-free context, it is possible to demonstrate that, although not every homogeneous and isotropic thermoelastic material can sustain thermoelastic antiplane shear motions, all thermoelastic materials defined in compliance with (2.4.1)–(2.4.3) and (2.4.10) are capable of doing so. It is easily shown that for such materials the local balance equations $(2.1.11)_{2,3,4,5}$ reduce, in the absence of body forces and heat supplies, to

$$(M(\gamma, \theta)u_{,\alpha})_{,\alpha} = \rho\ddot{u} \quad \text{on} \quad \mathcal{X} \setminus \Gamma,$$

$$(\tilde{k}(\gamma, \theta)\theta_{,\alpha})_{,\alpha} + M_\theta(\gamma, \theta)\theta u_{,\alpha}\dot{u}_{,\alpha} = \rho\tilde{c}(\gamma, \theta)\dot{\theta} \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \tag{2.6.2}$$

where $\mathcal{X}$ is given by $\mathcal{D} \times \mathcal{T}$, $\mathcal{D}$ is a generic cross section of $\mathcal{R}$, and $\Gamma = \{(x_1, x_2, t) | (x_1, x_2) \in C_t, t \in \mathcal{T}\}$ with $C_t = \mathcal{D} \cap S_t$ at each t in $\mathcal{T}$. See FOSDICK &

SERRIN [8] and FOSDICK & KAO [7] for a general discussion of the circumstances under which the local balance equations $(2.1.11)_{2,3}$ reduce to a single scalar equation. In (2.4.2) M is the secant modulus in shear as defined in (2.4.11) and $\gamma : \mathcal{X} \setminus \Gamma \to I\!\!R$ is the *shear strain field* given by

$$\gamma(x_1, x_2, t) = \sqrt{u_{,\alpha}(x_1, x_2, t)u_{,\alpha}(x_1, x_2, t)} \quad \forall(x_1, x_2, t) \in \mathcal{X} \setminus \Gamma. \qquad (2.6.3)$$

The jump conditions $(2.1.13)_{2,3}$ reduce, for a thermoelastic material of the type at hand subjected to antiplane shear, to

$$[\![M(\gamma, \theta)u_{,\alpha} n_\alpha]\!] + \rho V_n[\![\dot{u}]\!] = 0 \quad \text{on} \quad \Gamma,$$

$$[\![\tilde{k}(\gamma, \theta)\theta_{,\alpha} n_\alpha]\!] + \rho V_n \theta[\![\eta]\!] \cdot fV_n = 0 \quad \text{on} \quad \Gamma, \qquad (2.6.4)$$

where $\Gamma = \{(\mathbf{x}, t) | \mathbf{x} \in C_t, t \in \mathcal{T}\}$, $\mathbf{n}(\cdot, t) : C_t \to \mathcal{N}$ is a unit normal to C_t, the nominal entropy per unit mass $\eta : \mathcal{X} \setminus \Gamma \to I\!\!R$ is given, from $(2.3.1)_3$, (2.4.1) and (2.4.3), by

$$\eta = -\tilde{\psi}_\theta(3 + \gamma^2, \theta) = -\frac{1}{\rho} \int_0^\gamma \tau_\theta(\kappa, \theta)\,d\kappa \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \qquad (2.6.5)$$

and $f : \Gamma \to I\!\!R$ is the driving traction introduced in Section 2.3. The kinematic jump condition (2.1.16) becomes

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma, \quad [\![\theta]\!] = 0 \quad \text{on} \quad \Gamma. \qquad (2.6.6)$$

It is also readily shown that the driving traction f for a thermoelastic material defined via (2.4.1)–(2.4.3) and (2.4.10) subjected to an antiplane shear deformation involving a discontinuity in the gradient of the out-of-plane displacement across a moving curve C_t is given by

$$f = \int_{\gamma^-}^{\gamma^+} \tau(\kappa, \theta)\,d\kappa - \langle\!\langle M(\gamma, \theta)u_{,\alpha} \rangle\!\rangle[\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \qquad (2.6.7)$$

Recall that the jump condition $(2.1.13)_4$, or, equivalently, $(2.2.7)$ is satisfied constitutively by requiring that the kinetic response function be admissible.

With reference to (2.1.14), (2.1.15) and (2.2.6) it is easily demonstrated that, in the absence of inertial effects, (2.4.2) is replaced by

$$\big(M(\gamma,\theta)u_{,\alpha}\big)_{,\alpha} = 0 \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \tag{2.6.8}$$

while (2.4.4) becomes

$$[\![M(\gamma,\theta)u_{,\alpha}\, n_\alpha]\!] = 0 \quad \text{on} \quad \Gamma, \tag{2.6.9}$$

and (2.6.7) reduces to

$$f = \int_{\underset{-}{\gamma}}^{\overset{+}{\gamma}} \tau(\kappa,\theta)\, d\kappa - M(\overset{\pm}{\gamma},\theta)\overset{\pm}{u}_{,\alpha}\, [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \tag{2.6.10}$$

Observe that, within the context of a thermoelastic antiplane shear deformation of the type described above, no generality is lost by focussing exclusively upon the motion on a cross-section $\mathcal{D}$ of the cylinder $\mathcal{R}$ and the dynamics of the curve $C_t = \mathcal{D} \cap S_t$. In the following, curves C_t across which the gradient of the out-of-plane displacement field $u(\cdot,\cdot,t)$ and, perhaps, the out-of-plane velocity field $\dot{u}(\cdot,\cdot,t)$, the entropy field $\eta(\cdot,\cdot,t)$, and the gradient of the absolute temperature field $\theta(\cdot,\cdot,t)$ jumps, at some instant t in $\mathcal{T}$, and which segregate the high and low strain phases of the material at hand will, therefore, be referred to as *phase boundaries*.

3. LINEAR STABILITY OF A PROCESS INVOLVING A STEADILY MOVING PLANAR PHASE BOUNDARY IN A THREE-PHASE THERMOELASTIC MATERIAL

3.1. Description of the base process. Suppose that $\mathcal{B}$ is composed of a three-phase thermoelastic material and that the cylinder $\mathcal{R}$ degenerates so as to occupy all of $I\!\!R^3$. Let the rectangular Cartesian frame X be as in Section 2.4. Consider a thermoelastic antiplane shear motion on the time interval $(-\infty, 0)$ with an out-of-plane displacement field $u_0(\cdot, t) : I\!\!R \to I\!\!R$ given by

$$u_0(x_1, t) = \begin{cases} \gamma_l x_1 + v_l t & \text{if} \quad x_1 < v_0 t, \\ \gamma_r x_1 + v_r t & \text{if} \quad x_1 > v_0 t, \end{cases} \tag{3.1.1}$$

for each t in $(-\infty, 0)$, and an absolute temperature field θ_0 which is constant on $\mathcal{R} \times (-\infty, 0)$ and satisfies

$$\theta_0 \in (\theta_m, \theta_M), \tag{3.1.2}$$

where the shear strain-temperature pairs (γ_l, θ_0) and (γ_r, θ_0) satisfy one of the following

$$((\gamma_l, \theta_0), (\gamma_r, \theta_0)) \in A_3 \times A_1, \quad ((\gamma_l, \theta_0), (\gamma_r, \theta_0)) \in A_1 \times A_3. \tag{3.1.3}$$

Observe that the process described by (3.1.1)–(3.1.3) is *isothermal.*

Since one of (3.1.3) must hold, there is no loss in generality incurred by assuming that the base interface normal velocity v_0 is non-negative; that is,

$$v_0 \geq 0. \tag{3.1.4}$$

It is clear that u_0 and θ_0 satisfy the differential equations in (2.6.2) on the set $(I\!\!R^2 \times (-\infty, 0)) \setminus \Gamma_0$ with Γ_0 given by $\{(x_1, x_2, t) | (x_1, x_2) \in A_t, t \in (-\infty, 0)\}$ and $A_t = \{(x_1, x_2) | x_1 = v_0 t, x_2 \in I\!\!R\}$ for each t in $(-\infty, 0)$. The moving line A_t is, for each t in $(-\infty, o)$, a phase boundary.

Assume, in order to comply with the jump conditions in (2.6.4) and (2.6.6) on Γ_0, that the constants γ_l, γ_r, v_l, v_r, and v_0 associated with (3.1.1)–(3.1.3) are restricted to satisfy the following equations:

$$v_r - v_l + v_0(\gamma_r - \gamma_l) = 0,$$

$$\tau(\gamma_r, \theta_0) - \tau(\gamma_l, \theta_0) + \rho v_0(v_r - v_l) = 0, \qquad (3.1.5)$$

$$v_0(f_0 + \rho\theta_0(\eta_r - \eta_l)) = 0.$$

In (3.1.5) the base driving traction f_0 is given, with the aid of (2.6.7), by

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma, \theta_0)\, d\gamma - \tfrac{1}{2}(\tau(\gamma_r, \theta_0) + \tau(\gamma_l, \theta_0))(\gamma_r - \gamma_l), \qquad (3.1.6)$$

and the constants η_r and η_l are given in terms of γ_r, γ_l and θ_0 via the shear stress response function τ as follows

$$\eta_r = -\frac{1}{\rho}\int_0^{\gamma_r} \tau_\theta(\kappa, \theta_0)\, d\kappa, \quad \eta_l = -\frac{1}{\rho}\int_0^{\gamma_l} \tau_\theta(\kappa, \theta_0)\, d\kappa. \qquad (3.1.7)$$

Observe, as a consequence of (3.1.2) and (2.2.7), that f_0 must satisfy

$$f_0 \geq 0. \qquad (3.1.8)$$

Assume that v_0 complies with the inequality

$$v_0 < \min\left\{ \sqrt{\tau'(\gamma_l, \theta_0)/\rho}, \sqrt{\tau'(\gamma_r, \theta_0)/\rho} \right\}, \qquad (3.1.9)$$

so that the normal velocity of the phase boundary in the base process is *locally subsonic*. It is then permissible[2] to impose a kinetic relation in the form (2.3.8)

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[2] See ABEYARATNE & KKNOWLES [2].

or (2.3.9) on Γ_0 and require that the parameters γ_l, γ_r, v_l, v_r, and v_0 satisfy one of

$$v_0 = \tilde{V}(\tfrac{f_0}{\theta_0}, \theta_0), \quad f_0 = \theta_0 \tilde{\varphi}(v_0, \theta_0), \tag{3.1.10}$$

depending, respectively, upon whether a kinetic relation is provided in the form (2.5.1) or (2.5.2).

In a coordinate frame moving with the phase boundary, the base process described involves a piecewise homogeneous shear strain field and a homogeneous temperature field. If (γ_l, θ_0) and (γ_r, θ_0) are consistent with $(3.1.3)_1$ then the base process is one wherein the high strain elliptic phase of the material at hand *grows* at the expense of the low strain elliptic phase at constant temperature; whereas, if (γ_l, θ_0) and (γ_r, θ_0) comply with $(3.1.3)_2$ then the base process is such that the low strain elliptic phase of the material at hand *grows* at the expense of the high strain elliptic phase at constant temperature. In either case the discontinuity involved is, for the duration of the motion, a *normal* phase boundary—that is, the angle between the limiting values of the gradient of the out-of-plane displacement field on either side of the phase boundary is zero at every point of the phase boundary over the time interval $(-\infty, 0)$.

The constant *latent heat of transformation*—ℓ_0—associated with the thermoelastic process described by (3.1.1)–(3.1.3) is defined by

$$\ell_0 = \rho\theta_0(\eta_l - \eta_r) - f_0 = \theta_0 \int_{\gamma_l}^{\gamma_r} \tau_\theta(\kappa, \theta_0) \, d\kappa - f_0. \tag{3.1.11}$$

From $(3.1.5)_3$ it is clear that ℓ_0 must be zero if $v_0 = 0$—which agrees with the intuitive notion that the heat given off in the process of transformation must be zero in the absence of *heat flux*. Recall from Section 2.5 that, under the present assumption that the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ which is provided is continuous, $v_0 = 0$ if and only if $f_0 = 0$. Hence, when $v_0 = 0$, the latent heat

of transformation simplifies to

$$\ell_0 = \rho\theta_0(\eta_l - \eta_r) = \theta_0 \int\limits_{\gamma_l}^{\gamma_r} \tau_\theta(\kappa, \theta_0)\, d\kappa. \tag{3.1.12}$$

Observe, however, that $(3.1.5)_3$ is satisfied for any real value of ℓ_0 when $v_0 = 0$.

Suppose, in addition to all the above, that the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that its derivative is non-zero at the base driving traction f_0, that is, assume that one of the following, as is appropriate to the specification of a kinetic relation in the form of either (2.5.1) or (2.5.2), must hold:

$$\tilde{V}_\Phi(\tfrac{f_0}{\theta_0}, \theta_0) \neq 0, \quad \tilde{\varphi}_V(v_0, \theta_0) \neq 0, \tag{3.1.13}$$

This assumption is made in order to preclude the necessity of going to higher order in the context of the forthcoming linear stability analysis. See Figure 3 and Figure 4 for schematic graphs of smooth admissible kinetic response functions $\tilde{V}(\cdot, \theta_0)$ and $\tilde{\varphi}(\cdot, \theta_0)$ which satisfy (3.1.13).

When inertial effects are ignored it is clear that u_0 as defined in (3.1.1) also satisfies the field equation in (2.6.9) on $(\mathbb{R}^2 \times (-\infty, 0)) \setminus \Gamma_0$. Equations $(3.1.5)_{1,3}$ are, in this context, still sufficient to satisfy (2.6.8) and $(2.6.2)_2$ on Γ_0. In place of $(3.1.5)_2$, the constants γ_l, γ_r, v_l, v_r, and v_0 must, however, satisfy

$$\tau(\gamma_r, \theta_0) - \tau(\gamma_l, \theta_0) = 0, \tag{3.1.14}$$

in order for the jump condition in (2.6.9) to hold on Γ_0. Although the expression for the base driving traction f_0 given in (3.1.6) remains valid in the inertia-free setting, (3.1.14) can in this case be used so that it simplifies to read

$$f_0 = \int\limits_{\gamma_l}^{\gamma_r} \tau(\gamma, \theta_0)\, d\gamma - \tau_*(\gamma_r - \gamma_l), \tag{3.1.15}$$

where $\tau_* = \tau(\gamma_l, \theta_0) = \tau(\gamma_r, \theta_0)$.

Given a shear stress response function τ which describes a particular three-phase thermoelastic material and an arbitrary kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ which describes the dynamics of phase boundaries which may occur therein, there may not, in general, exist constants γ_l, γ_r, v_l, v_r, and v_0 which satisfy one of $(3.1.3)_1$ or $(3.1.3)_2$ and are consistent with the restrictions embodied by $(3.1.5)$, $(3.1.9)$, $(3.1.10)_1$ or $(3.1.10)_2$, and $(3.1.13)_1$ or $(3.1.13)_2$, or, in the inertia-free case, $(3.1.5)_1$, $(3.1.13)_1$ or $(3.1.13)_2$, $(3.1.9)$, $(3.1.10)_1$ or $(3.1.10)_2$, $(3.1.14)$ and $(3.1.15)$. Within the context of this investigation it will be assumed, however, that $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that a non-trivial base process exists.

3.2. Perturbation of the base process. Suppose that at the instant $t = 0$ the out-of-plane displacement and velocity fields, the absolute temperature field and the configuration of the phase boundary associated with the thermoelastic process specified in Section 3.1 are subjected to a perturbation. Let this perturbation be such that the phase boundary can be, at $t = 0+$, described by the graph C_0 of a continuous function $h : \mathbb{R} \to \mathbb{R}$ of the x_2–coordinate, and segregates elliptic phases of the three-phase material at hand in a sense consistent with that which was present for t in $(-\infty, 0)$. Let the out-of-plane displacement field, out-of-plane velocity field, and absolute temperature field linked to this perturbation be given, respectively, by a once continuously differentiable function $\eta : \mathbb{R}^2 \to \mathbb{R}$, a continuous function $\varpi : \mathbb{R}^2 \to \mathbb{R}$, and a continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$. Assume that h, η, ϖ and ϕ represent small deviations, in some appropriate sense, from their counterparts in the base process. In particular, suppose that h, η, $\eta_{,\alpha}$, ϖ, and ϕ are all square integrable on their domains of definition. Require, furthermore, that the components of the gradient of η allow the satisfaction of the decay condition

$$\lim_{x_1^2 + x_2^2 \to \infty} \eta_{,\alpha}(x_1, x_2)\eta_{,\alpha}(x_1, x_2) = 0, \qquad (3.2.1)$$

while ϖ and ϕ comply with the following decay conditions

$$\lim_{x_1^2+x_2^2\to\infty} \varpi(x_1,x_2) = 0, \qquad \lim_{x_1^2+x_2^2\to\infty} \phi(x_1,x_2) = 0, \qquad (3.2.2)$$

so that the disturbance is *localized* in a neighborhood of the phase boundary associated with the base state at $t = 0$.

The perturbation at $t = 0$ will initiate a new process involving an out-of-plane displacement field $u : \mathbb{R}^2 \times \overline{\mathbb{R}}_+ \to \mathbb{R}$ and an absolute temperature field $\theta : \mathbb{R}^2 \times \overline{\mathbb{R}}_+ \to \mathbb{R}$ which are, in general, *weak solutions* of the field equations (2.6.2) and satisfy the jump conditions in (2.6.4) and (2.6.6) at all discontinuities in their gradients, the kinetic relation (2.5.1) or (2.5.2) at all phase boundaries, and the initial conditions

$$u(\cdot,\cdot,0+) = u_0(\cdot,0+) + \eta \quad \text{on} \quad \mathbb{R}^2,$$

$$\dot{u}(\cdot,\cdot,0+) = \dot{u}_0(\cdot,0+) + \varpi \quad \text{on} \quad \mathbb{R}^2, \qquad (3.2.3)$$

$$\theta(\cdot,\cdot,0+) = \theta_0 + \phi \quad \text{on} \quad \mathbb{R}^2.$$

Since the perturbation is small, assume that, the subsequent process involves only a single phase boundary $C_t = \{(x_1,x_2,t) \mid x_1 = \varsigma(x_2,t), x_2 \in \mathbb{R}\}$ for each t in $\mathbb{R}_+$, with $\varsigma : \mathbb{R} \times \overline{\mathbb{R}}_+ \to \mathbb{R}$ continuously differentiable on its domain of definition and defined so that it is in accord with the initial condition

$$\varsigma(\cdot,0+) = h \quad \text{on} \quad \mathbb{R}. \qquad (3.2.4)$$

With the intent of linearizing the field equations in (2.6.2) about the base process, write, for each t in $\overline{\mathbb{R}}_+$,

$$u(x_1,x_2,t) = u_0(x_1,t) + w(x_1,x_2,t) \quad \forall(x_1,x_2) \in \mathcal{D} \setminus C_t,$$

$$\theta(x_1,x_2,t) = \theta_0 + T(x_1,x_2,t) \quad \forall(x_1,x_2) \in \mathcal{D} \setminus C_t, \qquad (3.2.5)$$

where w and its derivatives and T are assumed to represent small departures from the relevent quantities in the base process. Assume that the components of the gradient of w satisfy the following limits

$$\lim_{x_1 \to \pm\infty} w_{,1}(x_1, \cdot, \cdot) = 0 \quad \text{on} \quad I\!R \times I\!R_+,$$

$$\lim_{x_2 \to \pm\infty} w_{,2}(\cdot, x_2, \cdot) = 0 \quad \text{on} \quad I\!R \times I\!R_+,$$

(3.2.6)

and also that T complies with the limit

$$\lim_{x_1^2 + x_2^2 \to \infty} T(x_1, x_2, \cdot) = 0 \quad \text{on} \quad \bar{I\!R}_+.$$

(3.2.7)

From (3.2.3) and (3.2.5) it is clear, moreover, that—when inertial effects are not ignored—the increment w to the out-of-plane displacement field must satisfy the following initial conditions:

$$w(\cdot, \cdot, 0+) = \eta \quad \text{on} \quad I\!R^2,$$

$$\dot{w}(\cdot, \cdot, 0+) = \varpi \quad \text{on} \quad I\!R^2.$$

(3.2.8)

It is important to emphasize that these can not be imposed in the inertia-free setting.

Also, the increment T to the absolute temperature field must satisfy the following initial condition

$$T(\cdot, \cdot, 0+) = \phi \quad \text{on} \quad I\!R^2.$$

(3.2.9)

Next, define $s : I\!R \times I\!R_+ \to I\!R$, the correction to the interface position due to the perturbation, via

$$\varsigma(\cdot, t) = v_0 t + s(\cdot, t) \quad \text{on} \quad I\!R \quad \forall t \in I\!R_+.$$

(3.2.10)

Note, from (3.2.4) that the increment s to the phase boundary position must satisfy the initial condition

$$s(\cdot, 0+) = h \quad \text{on} \quad I\!\!R. \tag{3.2.11}$$

Observe that the unit normal vectors $n_\pm(\cdot, t) : I\!\!R \to \mathcal{N}$ to C_t are given by

$$n_\pm(\cdot, t) = \pm \frac{e_1 - s_{,2}(\cdot, t)e_2}{\sqrt{1 + s_{,2}^2(\cdot, t)}} \quad \text{on} \quad I\!\!R \quad \forall t \in I\!\!R_+. \tag{3.2.12}$$

For the remainder of this work, choose the unit normal vector associated with the plus sign in (3.2.12) and drop this sign when referring to it. The normal velocity $V_n(\cdot, t) : I\!\!R \to I\!\!R$ of C_t is, thus, given, for each t in $I\!\!R_+$, by

$$V_n(\cdot, t) = \frac{v_0 + \dot{s}(\cdot, t)}{\sqrt{1 + s_{,2}^2(\cdot, t)}} \quad \text{on} \quad I\!\!R \quad \forall t \in I\!\!R_+. \tag{3.2.13}$$

3.3. Linearization of the field equations associated with the process initiated by the perturbation.

Let $\mathcal{D}_t^l$ and $\mathcal{D}_t^r$ denote, for each t in $I\!\!R_+$, plane sets defined as shown below:

$$\mathcal{D}_t^l = \{(x_1, x_2) | x_1 \leq \varsigma(x_2, t)\}, \quad \mathcal{D}_t^r = I\!\!R^2 \setminus \overset{\circ}{\mathcal{D}}_t^l. \tag{3.3.1}$$

Let $\mathcal{X}_l$ and $\mathcal{X}_r$ be given, in turn, by

$$\mathcal{X}_l = \{(x_1, x_2, t) | (x_1, x_2) \in \mathcal{D}_t^l, t \in I\!\!R_+\}, \tag{3.3.2}$$

and

$$\mathcal{X}_r = \{(x_1, x_2, t) | (x_1, x_2) \in \mathcal{D}_t^r, t \in I\!\!R_+\}. \tag{3.3.3}$$

The displacement equations of motion which hold on $\overset{\circ}{\mathcal{X}}_l$ and $\overset{\circ}{\mathcal{X}}_r$ can be obtained following FRIED [9] and are given, in turn, by

$$a_l^2 w_{,11} + b_l^2 w_{,22} = \ddot{w}, $$
$$a_r^2 w_{,11} + b_r^2 w_{,22} = \ddot{w}, \tag{3.3.4}$$

where the positive constants a_l, b_l, and a_r, b_r are defined as follows:

$$a_l = \sqrt{\tau_\gamma(\gamma_l,\theta_0)/\rho}, \quad b_l = \sqrt{M(\gamma_l,\theta_0)/\rho},$$
$$a_r = \sqrt{\tau_\gamma(\gamma_r,\theta_0)/\rho}, \quad b_r = \sqrt{M(\gamma_r,\theta_0)/\rho}. \tag{3.3.5}$$

In writing (3.3.5), the positivity of $\tau_\gamma(\gamma_l,\theta_0)$ and $\tau_\gamma(\gamma_r,\theta_0)$—which are results of whichever of $(3.1.3)_{1,2}$ is appropriate, and of $M(\gamma_l,\theta_0)$ and $M(\gamma_r,\theta_0)$—which follow from (2.4.13), have been used.

The energy equations which hold on $\overset{\circ}{\mathcal{X}}_l$ and $\overset{\circ}{\mathcal{X}}_r$ can be obtained by linearizing the partial differential equation $(2.6.2)_2$ about (γ_l,θ_0) and (γ_r,θ_0), respectively. Turn, now, to the derivation of the linearized energy equation which holds on $\overset{\circ}{\mathcal{X}}_l$. It is easy to show, following [9], that

$$\gamma \cong \gamma_l + w,_1 \quad \text{on} \quad \overset{\circ}{\mathcal{X}}_l. \tag{3.3.6}$$

With the aid of $(3.2.5)_2$ and Taylor's theorem the relation (3.3.6) leads to the following expansions:

$$\tilde{k}(\gamma,\theta) \cong \tilde{k}(\gamma_l,\theta_0) + \tilde{k}_\gamma(\gamma_l,\theta_0)w,_1 + \tilde{k}_\theta(\gamma_l,\theta_0)T \quad \text{on} \quad \overset{\circ}{\mathcal{X}}_l,$$
$$\tilde{c}(\gamma,\theta) \cong \tilde{c}(\gamma_l,\theta_0) + \tilde{c}_\gamma(\gamma_l,\theta_0)w,_1 + \tilde{c}_\theta(\gamma_l,\theta_0)T \quad \text{on} \quad \overset{\circ}{\mathcal{X}}_l, \tag{3.3.7}$$
$$M_\theta(\gamma,\theta) \cong M_\theta(\gamma_l,\theta_0) + M_{\gamma\theta}(\gamma_l,\theta_0)w,_1 + M_{\theta\theta}(\gamma_l,\theta_0)T \quad \text{on} \quad \overset{\circ}{\mathcal{X}}_l.$$

Next, using (3.2.5) and $(3.3.7)_{1,2}$ in the left hand side of the partial differential equation in $(2.6.2)_2$ gives

$$(\tilde{k}(\gamma,\theta)\theta,_\alpha),_\alpha + M_\theta(\gamma,\theta)\theta u,_\alpha \dot{u},_\alpha \cong$$
$$\tilde{k}(\gamma_l,\theta_0)T,_{\alpha\alpha} + \theta_0\tau_\theta(\gamma_l,\theta_0)\dot{w},_1 \quad \text{on} \quad \overset{\circ}{\mathcal{X}}_l, \tag{3.3.8}$$

while using (3.2.5) and $(3.2.7)_3$ in the right-hand-side of the same equation gives

$$\rho\tilde{c}(\gamma,\theta)\dot{\theta} \cong \rho\tilde{c}(\gamma_l,\theta_0)\dot{T} \quad \text{on} \quad \overset{\circ}{\mathcal{X}}_l. \tag{3.3.9}$$

The linearized energy equation which holds on $\overset{\circ}{\mathcal{X}}_l$ is, thus, from (3.2.5), (3.3.8) and (3.3.9), given by

$$\alpha_l T,_{\alpha\alpha} = \dot{T} + \beta_l \dot{w},_1 , \tag{3.3.10}$$

where the positive constant α_l and the real constant β_l are defined by

$$\alpha_l = \frac{\tilde{k}(\gamma_l, \theta_0)}{\rho \tilde{c}(\gamma_l, \theta_0)}, \quad \beta_l = -\frac{\theta_0 \tau_\theta(\gamma_l, \theta_0)}{\rho \tilde{c}(\gamma_l, \theta_0)}. \tag{3.3.11}$$

Similarly, the linearized energy equation which holds on $\overset{\circ}{\mathcal{X}}_r$ is

$$\alpha_r T,_{\alpha\alpha} = \dot{T} + \beta_r \dot{w},_1 , \tag{3.3.12}$$

where the positive constant α_r and the real constant β_r are defined by

$$\alpha_r = \frac{\tilde{k}(\gamma_r, \theta_0)}{\rho \tilde{c}(\gamma_r, \theta_0)}, \quad \beta_r = -\frac{\theta_0 \tau_\theta(\gamma_r, \theta_0)}{\rho \tilde{c}(\gamma_r, \theta_0)}. \tag{3.3.13}$$

From (2.6.9) it is clear that, in the inertia-free setting, the displacement equations of motion (3.3.4) are supplanted by

$$\begin{aligned}
a_l^2 w,_{11} + b_l^2 w,_{22} &= 0, \\
a_r^2 w,_{11} + b_r^2 w,_{22} &= 0,
\end{aligned} \tag{3.3.14}$$

which hold, repectively, on $\overset{\circ}{\mathcal{X}}_l$ and $\overset{\circ}{\mathcal{X}}_r$.

3.4. Linearization of the jump conditions and kinetic relation associated with the process initiated by the perturbation.

Since the set $\Gamma = \{(x_1, x_2, t) | (x_1, x_2) \in C_t, t \in I\!\!R_+\}$ represents the post-disturbance trajectory of the phase boundary, the jump conditions in (2.6.4) and (2.6.6) and the kinetic relation in (2.5.1) or (2.5.2)—with V_n and f given, respectively, by (3.2.13) and (2.6.7)—must hold on it. Assume, henceforth, that the function s introduced via (3.2.10) and its derivatives are small in the same sense that w and T are small.

Note, first, that this assumption implies, using (3.2.12) and (3.2.13), the following approximations for $\mathbf{n}$ and V_n on Γ:

$$\mathbf{n} \cong \mathbf{e}_1 - s_{,2}\,\mathbf{e}_2 \quad \text{on} \quad \Gamma,$$

$$V_n \cong v_0 + \dot{s} \quad \text{on} \quad \Gamma. \tag{3.4.1}$$

It is easy to show, following FRIED [9], that the linearized form of the jump condition $(2.6.4)_1$ is as follows

$$(a_r^2 - v_0^2)w_{,1}\,(v_0t+, x_2, t) - (a_l^2 - v_0^2)w_{,1}\,(v_0t-, x_2, t)$$
$$= 2v_0(\gamma_r - \gamma_l)\dot{s}(x_2, t) \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.4.2}$$

while that of $(2.6.6)_1$ is

$$w(v_0t+, x_2, t) - w(v_0t-, x_2, t) = (\gamma_l - \gamma_r)s(x_2, t) \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+. \tag{3.4.3}$$

Linearization of the jump condition $(2.6.6)_2$ gives, next, since no heat flux is present in the base process (3.1.1)–(3.1.3),

$$T(v_0t+, x_2, t) - T(v_0t-, x_2, t) = 0 \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.4.4}$$

so that the increment T to the absolute temperature field is continuous across the phase boundary in the post perturbation process.

The driving traction f can be linearized in a manner analagous to that displayed in [9] to give, with the aid of (3.4.4),

$$f(x_2, t) \cong f_0 + \tfrac{1}{2}\rho(\gamma_l - \gamma_r)((a_r^2 - v_0^2)w_{,1}\,(v_0t+, x_2, t) + (a_l^2 - v_0^2)w_{,1}\,(v_0t-, x_2, t))$$
$$+ \left(\rho(\eta_l - \eta_r) + \tfrac{1}{2}(\gamma_l - \gamma_r)(\tau_\theta(\gamma_r, \theta_0) + \tau_\theta(\gamma_l, \theta_0))\right)T(v_0t, x_2, t)$$
$$\forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.4.5}$$

where the base driving traction f_0 is given by (3.1.6).

Turn, now, to the linearization of the energy jump condition $(2.6.4)_2$. From $(3.3.7)_1$, $(3.4.1)_1$ and $(3.2.5)_2$ it is clear that the first term on the left-hand-side of the energy balance jump condition $(2.6.4)_2$ linearizes as follows:

$$[\tilde{k}(\gamma(\zeta(x_2,t),x_2,t),\theta(\zeta(x_2,t),x_2,t))\theta_{,\alpha}(\zeta(x_2,t),x_2,t)n_\alpha(x_2,t)]$$

$$\cong \tilde{k}(\gamma_r,\theta_0)T_{,1}(v_0t+,x_2,t) - \tilde{k}(\gamma_l,\theta_0)T_{,1}(v_0t-,x_2,t)$$

$$\forall(x_2,t) \in I\!\!R \times I\!\!R_+. \tag{3.4.6}$$

Furthermore, (2.6.5), $(2.6.6)_2$, (3.4.1), (3.4.4) and (3.4.5) yield the following linearization of the remaining two terms on the left-hand-side of the energy balance jump condition $(2.6.4)_2$:

$$\left(\rho\theta(\zeta(x_2,t),x_2,t)[\eta(\zeta(x_2,t),x_2,t)] + f(x_2,t)\right)V_n(x_2,t)$$

$$\cong \rho v_0(\tilde{c}(\gamma_r,\theta_0) - \tilde{c}(\gamma_l,\theta_0))T(v_0t,x_2,t)$$

$$+ \tfrac{1}{2}v_0(\gamma_l - \gamma_r)(\tau_\theta(\gamma_r,\theta_0) + \tau_\theta(\gamma_l,\theta_0)))T(v_0t,x_2,t)$$

$$+ \tfrac{1}{2}\rho v_0(\gamma_l - \gamma_r)((a_r^2 - v_0^2)w_{,1}(v_0t+,x_2,t) + (a_l^2 - v_0^2)w_{,1}(v_0t-,x_2,t))$$

$$- v_0\theta_0(\tau_\theta(\gamma_r,\theta_0)w_{,1}(v_0t+,x_2,t) - \tau_\theta(\gamma_l,\theta_0)w_{,1}(v_0t-,x_2,t))$$

$$- \ell_0\dot{s}(x_2,t) \quad \forall(x_2,t) \in I\!\!R \times I\!\!R_+. \tag{3.4.7}$$

Together, (3.4.6) and (3.4.7) give the following expression for the linearization of the energy jump condition $(2.6.4)_2$:

$$0 = \tilde{k}(\gamma_r,\theta_0)T_{,1}(v_0t+,x_2,t) - \tilde{k}(\gamma_l,\theta_0)T_{,1}(v_0t-,x_2,t)$$

$$+ \rho v_0(\tilde{c}(\gamma_r,\theta_0) - \tilde{c}(\gamma_l,\theta_0))T(v_0t,x_2,t)$$

$$+ \tfrac{1}{2}v_0(\gamma_l - \gamma_r)(\tau_\theta(\gamma_r,\theta_0) + \tau_\theta(\gamma_l,\theta_0)))T(v_0t,x_2,t)$$

$$+ \tfrac{1}{2}\rho v_0(\gamma_l - \gamma_r)((a_r^2 - v_0^2)w_{,1}(v_0t+,x_2,t) + (a_l^2 - v_0^2)w_{,1}(v_0t-,x_2,t))$$

$$- v_0\theta_0(\tau_\theta(\gamma_r,\theta_0)w_{,1}(v_0t+,x_2,t) - \tau_\theta(\gamma_l,\theta_0)w_{,1}(v_0t-,x_2,t))$$

$$- \ell_0\dot{s}(x_2,t) \quad \forall(x_2,t) \in I\!\!R \times I\!\!R_+. \tag{3.4.8}$$

Linearization of the kinetic relation and use of whichever of (3.1.10) is appropriate in a manner completely analogous to that performed in [9] gives, with the aid of (3.4.4),

$$\dot{s}(x_2,t) = \frac{\gamma_l - \gamma_r}{2v_*}((a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) + (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t))$$

$$+ \frac{\gamma_l - \gamma_r}{2\rho v_*}(\tau_\theta(\gamma_r, \theta_0) + \tau_\theta(\gamma_l, \theta_0)))T(v_0t, x_2, t)$$

$$+ (v_\vartheta + \frac{\ell_0}{\rho v_* \theta_0})T(v_0t, x_2, t) \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.4.9}$$

where the constants v_* and v_ϑ are defined by either

$$v_* = \frac{\theta_0}{\rho \tilde{V}_\Phi(\frac{\ell_0}{\theta_0}, \theta_0)}, \quad v_\vartheta = \tilde{V}_\Theta(\frac{\ell_0}{\theta_0}, \theta_0), \tag{3.4.10}$$

if the kinetic relation is furnished in the form (2.5.1), or

$$v_* = \frac{\theta_0 \tilde{\varphi}_V(v_0, \theta_0)}{\rho}, \quad v_\vartheta = -\frac{\tilde{\varphi}_\theta(v_0, \theta_0)}{\tilde{\varphi}_V(v_0, \theta_0)}, \tag{3.4.11}$$

if the kinetic relation is supplied in the form (2.5.2). Note, from (3.1.13), that v_* is a real—but nonzero—constant, while v_ϑ is a real—and possibly zero—constant.

By virtue of the foregoing calculations it is crucial to note that, within the scope of the linearization, it is legitimate to enforce the partial differential equations in (3.3.4)$_1$ and (3.3.10) on the interiors of the set Ω_l defined by

$$\Omega_l = \{(x_1, x_2, t) | (x_1, x_2) \in \Pi_t^l, t \in \mathbb{R}_+\}, \tag{3.4.12}$$

with $\Pi_t^l = \{(x_1, x_2) | x_1 \leq v_o t, x_2 \in \mathbb{R}\}$ for each t in $\mathbb{R}_+$, instead of the set $\overset{\circ}{\mathcal{X}}_l$, and the partial differential equations in (3.3.4)$_2$ and (3.3.12) on the interior of the set Ω_r defined by

$$\Omega_r = \{(x_1, x_2, t) | (x_1, x_2) \in \Pi_t^r, t \in \mathbb{R}_+\}, \tag{3.4.13}$$

with $\Pi_t^r = \{(x_1, x_2) | x_1 \geq v_o t, x_2 \in I\!\!R\}$ for each t in $I\!\!R_+$, instead of the set $\overset{\circ}{\mathcal{X}}_r$. For the purposes of the forthcoming analysis it is useful to define a set I as follows:

$$I = \{(x_1, x_2, t) | x_1 = v_0 t, x_2 \in I\!\!R, t \in I\!\!R_+\}. \qquad (3.4.14)$$

In the inertia-free case it is readily shown that, while (3.4.3) and (3.4.4) continue to hold, (3.4.2) is replaced by

$$a_r^2 w_{,1}(v_0 t+, x_2, t) - a_l^2 w_{,1}(v_0 t-, x_2, t) = 0 \quad \forall (x_2, t) \in I\!\!R \times I\!\!R_+, \qquad (3.4.15)$$

(3.4.8) reduces to

$$\begin{aligned}
0 =& \tilde{k}(\gamma_r, \theta_0) T_{,1}(v_0 t+, x_2, t) - \tilde{k}(\gamma_l, \theta_0) T_{,1}(v_0 t-, x_2, t) \\
&+ \rho v_0(\tilde{c}(\gamma_r, \theta_0) - \tilde{c}(\gamma_l, \theta_0)) T(v_0 t, x_2, t) \\
&+ \tfrac{1}{2} v_0(\gamma_l - \gamma_r)(\tau_\theta(\gamma_r, \theta_0) + \tau_\theta(\gamma_l, \theta_0))) T(v_0 t, x_2, t) \\
&+ \tfrac{1}{2} \rho v_0(\gamma_l - \gamma_r)(a_r^2 w_{,1}(v_0 t+, x_2, t) + a_l^2 w_{,1}(v_0 t-, x_2, t)) \\
&- v_0 \theta_0(\tau_\theta(\gamma_r, \theta_0) w_{,1}(v_0 t+, x_2, t) - \tau_\theta(\gamma_l, \theta_0) w_{,1}(v_0 t-, x_2, t)) \\
&- \ell_0 \dot{s}(x_2, t) \quad \forall (x_2, t) \in I\!\!R \times I\!\!R_+, \qquad (3.4.16)
\end{aligned}$$

and (3.4.9) simplifies to read

$$\begin{aligned}
\dot{s}(x_2, t) =& \frac{\gamma_l - \gamma_r}{2 v_*}(a_r^2 w_{,1}(v_0 t+, x_2, t) + a_l^2 w_{,1}(v_0 t-, x_2, t)) \\
&+ \frac{\gamma_l - \gamma_r}{2 \rho v_*}(\tau_\theta(\gamma_r, \theta_0) + \tau_\theta(\gamma_l, \theta_0))) T(v_0 t, x_2, t) \\
&+ (v_\vartheta + \tfrac{\ell_0}{\rho v_* \theta_0}) T(v_0 t, x_2, t) \quad \forall (x_2, t) \in I\!\!R \times I\!\!R_+. \qquad (3.4.17)
\end{aligned}$$

Finally, remarks analogous to those made regarding the enforcement of the partial differential equations in $(3.3.4)_1$ and $(3.3.4)_2$ on $\overset{\circ}{\Omega}_l$ and $\overset{\circ}{\Omega}_r$ apply also to those in $(3.3.14)_1$ and $(3.3.14)_2$.

3.5. Specialization of the base process and the associated linearized description of the post perturbation process.

Suppose, henceforth, that the base process described in Section 3.1 is restricted so that

$$\tau_\theta(\gamma_l, \theta_0) = \tau_\theta(\gamma_r, \theta_0) = 0; \qquad (3.5.1)$$

it is implicitly assumed that the shear stress response function τ allows for the possibility of (3.5.1). Observe that (3.5.1) requires that the *coefficients of thermoelastic coupling* in the low and high strain phases of the material at hand are, by $(3.3.11)_2$ and $(3.3.13)_2$, both identically zero. Although this assumption is made in order to simplify the forthcoming analysis, it is not inconsistent with the isochoric nature of the deformation under consideration. The linearized field equations, jump conditions, kinetic relation, initial conditions (where appropriate), and far field decay conditions satisfied by the increments w, T and s to the out-of-plane displacement field, absolute temperature field and the interface position are now listed in both the inertial and inertia-free cases.

In the inertial case, (3.3.4), (3.3.10) and (3.3.12) give the following linearized field equations

$$a_l^2 w_{,11} + b_l^2 w_{,22} = \ddot{w} \quad \text{on} \quad \mathring{\Omega}_l,$$

$$a_r^2 w_{,11} + b_r^2 w_{,22} = \ddot{w} \quad \text{on} \quad \mathring{\Omega}_r,$$

$$\alpha_l T_{,\alpha\alpha} = \dot{T} \quad \text{on} \quad \mathring{\Omega}_l, \qquad (3.5.2)$$

$$\alpha_r T_{,\alpha\alpha} = \dot{T} \quad \text{on} \quad \mathring{\Omega}_r.$$

In addition, from (3.4.2), (3.4.8), (3.4.3) and (3.4.4) the following jump conditions hold

$$[(a^2 - v_0^2)w_{,1}] = 2v_0(\gamma_r - \gamma_l)\dot{s} \quad \text{on} \quad I,$$

$$[kT_{,1}] + \rho v_0[c]T = \rho v_0(\gamma_r - \gamma_l)\langle\!\langle (a^2 - v_0^2)w_{,1} \rangle\!\rangle + \ell_0 \dot{s} \quad \text{on} \quad I,$$

$$[w] = (\gamma_l - \gamma_r)s \quad \text{on} \quad I, \qquad (3.5.3)$$

$$[\theta] = 0 \quad \text{on} \quad I,$$

where the constants a_+^2 and a_-^2 are given by

$$a_+^2 = a_r^2, \quad a_-^2 = a_l^2, \tag{3.5.4}$$

and $k_+ = k_r$, $k_- = k_l$, $c_+ = c_r$, and $c_- = c_l$ are defined via

$$k_+ = \tilde{k}(\gamma_r, \theta_0), \quad k_- = \tilde{k}(\gamma_l, \theta_0),$$

$$c_+ = \tilde{c}(\gamma_r, \theta_0), \quad c_- = \tilde{c}(\gamma_l, \theta_0). \tag{3.5.5}$$

Next, from (3.4.9) and (3.5.1) the following linearized kinetic relation holds:

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle (a^2 - v_0^2)w_{,1} \rangle\!\rangle + (v_\vartheta + \frac{\ell_0}{\rho v_* \theta_0})T \quad \text{on} \quad I. \tag{3.5.6}$$

Observe that, despite the restrictions imposed on the coefficients of thermoelastic coupling by (3.5.1), the corrections to the out-of-plane displacement and absolute temperature fields reamain coupled through $(3.5.3)_2$ and (3.5.6).

The initial conditions satisfied by w and s are, from (3.2.8), (3.2.9) and (3.2.11),

$$w(\cdot, \cdot, 0+) = \eta \quad \text{on} \quad I\!\!R^2,$$

$$\dot{w}(\cdot, \cdot, 0+) = \varpi \quad \text{on} \quad I\!\!R^2,$$

$$T(\cdot, \cdot, 0+) = \phi \quad \text{on} \quad I\!\!R^2, \tag{3.5.7}$$

$$s(\cdot, 0+) = h \quad \text{on} \quad I\!\!R.$$

Finally, from (3.2.6) and (3.2.7), it is assumed that, for each t in $I\!\!R_+$, the following far field decay conditions hold

$$\lim_{x_1 \to \pm\infty} w_{,1}(x_1, \cdot, t) = 0 \quad \text{on} \quad I\!\!R,$$

$$\lim_{x_2 \to \pm\infty} w_{,2}(\cdot, x_2, t) = 0 \quad \text{on} \quad I\!\!R, \tag{3.5.8}$$

$$\lim_{x_1^2 + x_2^2 \to \pm\infty} T(x_1, x_2, t) = 0.$$

In the inertia-free case, $(3.5.2)_{1,2}$ are replaced by

$$a_l^2 w_{,11} + b_l^2 w_{,22} = 0 \quad \text{on} \quad \mathring{\Omega}_l,$$

$$a_r^2 w_{,11} + b_r^2 w_{,22} = 0 \quad \text{on} \quad \mathring{\Omega}_r. \tag{3.5.9}$$

Furthermore, the jump condition $(3.5.3)_1$ is, by virtue of (3.4.15), replaced by

$$[a^2 w_{,1}] = 0 \quad \text{on} \quad I, \tag{3.5.10}$$

and $(3.5.3)_2$ is, from (3.4.16) and (3.5.1), supplanted by

$$[kT_{,1}] + \rho v_0 [c] T = \rho v_0 (\gamma_r - \gamma_l) \langle\!\langle a^2 w_{,1} \rangle\!\rangle + \ell_0 \dot{s} \quad \text{on} \quad I. \tag{3.5.11}$$

while $(3.5.3)_{3,4}$ continue to hold. Finally, the linearized kinetic relation (3.5.6) is, upon referring to (3.4.17) and (3.5.1), superceded by

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle a^2 w_{,1} \rangle\!\rangle + (v_\vartheta + \frac{\ell_0}{\rho v_* \theta_0}) T \quad \text{on} \quad I. \tag{3.5.12}$$

In the absence of inertial effects initial conditions cannot be given for the increments to the out-of-plane displacement and velocity fields w and $\dot{w}$; the initial condition $(3.5.7)_{3,4}$ pertaining to T and s still, however, continue to be applicable. The decay conditions (3.5.8) also still hold.

3.6. Normal mode analysis for a base process involving a static interface in the absence of inertia.

Suppose that v_0 in (3.1.1) is zero. Then the base process described by (3.1.1)–(3.1.3) is a piecewise homogeneous isothermal two-phase state involving a static planar interface. Recall, from Section 2.5, that when $v_0 = 0$ and the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is continuously differentiable on its domain of definition then $v_* \geq 0$. Since $v_* = 0$ is ruled out by whichever of (3.1.13) is appropriate and the corresponding expression $(3.4.10)_1$ or $(3.4.11)_1$, it is clear that—in the present context—$v_* > 0$. Consider, now, the

initial boundary value problem composed by (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$, (3.5.12), $(3.5.7)_{3,4}$ and (3.5.8). Note, since $v_0 = 0$, that (3.5.11) and (3.5.12) reduce with the aid of (3.1.12) to

$$[kT_{,1}] = \rho\theta_0(\eta_l - \eta_r)\dot{s} \quad \text{on} \quad I,$$

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*}\langle\!\langle a^2 w_{,1} \rangle\!\rangle + \frac{\eta_l - \eta_r}{v_*}T \quad \text{on} \quad I. \tag{3.6.1}$$

Observe that, by virtue of the linearization, the relevant partial differential equations, jump conditions and kinetic relation are all linear with constant coefficients; note, also, that the domains $\mathring{\Pi}_0^l$ and $\mathring{\Pi}_0^r$ are rectangular. It is therefore possible to find a solution to the linearized partial differential equations, jump conditions and kinetic relation in the form

$$w(x_1, x_2, t) = \begin{cases} W_l e^{+\xi_l x_1} e^{i\kappa x_2} e^{pt} & \forall(x_1, x_2, t) \in \mathring{\Pi}_0^l \times \mathbb{R}_+, \\ W_r e^{-\xi_r x_1} e^{i\kappa x_2} e^{pt} & \forall(x_1, x_2, t) \in \mathring{\Pi}_0^r \times \mathbb{R}_+, \end{cases}$$

$$T(x_1, x_2, t) = \begin{cases} \Theta_l e^{+\zeta_l x_1} e^{i\kappa x_2} e^{pt} & \forall(x_1, x_2, t) \in \mathring{\Pi}_0^l \times \mathbb{R}_+, \\ \Theta_r e^{-\zeta_r x_1} e^{i\kappa x_2} e^{pt} & \forall(x_1, x_2, t) \in \mathring{\Pi}_0^r \times \mathbb{R}_+, \end{cases} \tag{3.6.2}$$

$$s(x_2, t) = S e^{i\kappa x_2} e^{pt} \quad \forall(x_2, t) \in \mathbb{R} \times \mathbb{R}_+,$$

where the *amplitudes* W_l, W_r, Θ_l, Θ_r and S, *wave-numbers* ξ_l, ξ_r, ζ_l, ζ_r and κ, and *growth-rate* p are all constants. To comply with the decay conditions $(3.5.8)_{1,3}$ it is clear that $\Re(\xi_l)$, $\Re(\xi_r)$, $\Re(\zeta_l)$ and $\Re(\zeta_r)$ must all be positive. Although (3.6.2) is not, in general, consistent with neither the initial conditions $(3.5.7)_{3,4}$ which hold in the absence of inertial effects nor the decay conditions $(3.5.8)_{2,3}$, since ϕ and h are stipulated to be square integrable on $\mathbb{R}$, and hence can be represented as Fourier integrals—

$$\phi(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\phi}(x_1, \kappa)e^{i\kappa x_2}\, d\kappa \quad \forall(x_1, x_2) \in \mathbb{R}^2,$$

$$\tag{3.6.3}$$

$$h(x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{h}(\kappa)e^{i\kappa x_2}\, d\kappa \quad \forall x_2 \in \mathbb{R},$$

it is reasonable to expect that stability results can be obtained by a *normal-mode analysis*; such an analysis entails substitution of (3.6.2) into (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and (3.5.12) to determine the growth-rate p as a function of the complex wave-numbers ξ_l, ξ_r, ζ_l, ζ_r and the real wave-number κ. If there exists a complex growth-rate p with positive real part which arises as a solution to the aforementioned problem then the base process will be referred to as *linearly unstable*. Otherwise, the base process will be called *linearly stable*.

Observe that the amplitudes Θ_l, Θ_r and S and wave-numbers ζ_l, ζ_r and κ must be viewed as given for the normal mode analysis to prove effective in determining necessary and sufficient conditions, via the analysis of a *dispersion relation* like that performed in [9], for the linear instability of the base process with respect to arbitrary disturbances contained in the class of perturbations described in Section 3.2. Before proceeding, note, from $(3.5.3)_4$, that $\Theta_l = \Theta_r =: \Theta$. Now, substitution of (3.6.2) into (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and (3.5.12) yields the following relations

$$W_l = -\frac{\nu^2 |\kappa|}{a_l b_l (\gamma_l - \gamma_r)} S, \quad W_r = -\frac{\nu^2 |\kappa|}{a_r b_r (\gamma_r - \gamma_l)} S, \quad \Theta = -\frac{p G_0(\kappa, p)}{\eta_l - \eta_r} S,$$

$$\xi_l = \frac{b_l}{a_l}|\kappa|, \quad \xi_r = \frac{b_r}{a_r}|\kappa|, \quad \zeta_l = \sqrt{\kappa^2 + p/\alpha_l}, \quad \zeta_r = \sqrt{\kappa^2 + p/\alpha_r}, \qquad (3.6.4)$$

$$p + \frac{1}{v_*}(\nu^2 |\kappa| + p G_0(\kappa, p)) = 0,$$

where $G_0 : I\!\!R \times \mathbb{C} \to \mathbb{C}$ is given by

$$G_0(\kappa, p) = \frac{\theta_0 (\eta_l - \eta_r)^2}{c_l \sqrt{\alpha_l^2 \kappa^2 + \alpha_l p} + c_r \sqrt{\alpha_r^2 \kappa^2 + \alpha_r p}} \quad \forall (\kappa, p) \in I\!\!R \times \mathbb{C}, \qquad (3.6.5)$$

and the constant ν^2 is defined as follows:

$$\nu^2 = \frac{a_l b_l a_r b_r (\gamma_l - \gamma_r)^2}{a_l b_l + a_r b_r}. \qquad (3.6.6)$$

It is clear from $(3.6.4)_{3,6,7}$ that, for (3.6.2) to represent a solution to (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and (3.5.12), the amplitude Θ and the

wave-numbers ζ_l and ζ_r cannot be chosen independently of S and κ. Hence, the normal mode analysis is only of use in analyzing the linear stability of the base process at hand with respect to a certain class of perturbations; that is, it is only possible—via this analysis—to determine conditions necessary and sufficient for the instability of the base process with respect to a proper subset of the class of perturbations introduced in Section 3.2. To achieve such results it suffices to analyze the zero structure of the dispersion relation $(3.6.4)_8$ as a function of the growth-rate p for fixed values of the wave-number κ and the parameters γ_l, γ_r, v_0, θ_0, a_l, a_r, b_l, b_r, α_l, α_r, c_l, c_r, ρ and v_*. This is done below.

To comply with the restriction that $\Re(\zeta_l)$ and $\Re(\zeta_r)$ are both positive, the square roots which appear in the definition of G_0 are defined so that for p in $\mathbb{R}$,

$$
\begin{aligned}
\alpha_l^2 \kappa^2 + \alpha_l p > 0 &\implies \sqrt{\alpha_l^2 \kappa^2 + \alpha_l p} > 0 \quad \forall \kappa \in \mathbb{R}, \\
\alpha_r^2 \kappa^2 + \alpha_r p > 0 &\implies \sqrt{\alpha_r^2 \kappa^2 + \alpha_r p} > 0 \quad \forall \kappa \in \mathbb{R},
\end{aligned}
\tag{3.6.7}
$$

from which it is clear that for p in $\mathbb{C}$,

$$
\begin{aligned}
\Re(\alpha_l^2 \kappa^2 + \alpha_l p) > 0 &\implies \Re\left(\sqrt{\alpha_l^2 \kappa^2 + \alpha_l p}\right) > 0 \quad \forall \kappa \in \mathbb{R}, \\
\Re(\alpha_r^2 \kappa^2 + \alpha_r p) > 0 &\implies \Re\left(\sqrt{\alpha_r^2 \kappa^2 + \alpha_r p}\right) > 0 \quad \forall \kappa \in \mathbb{R}.
\end{aligned}
\tag{3.6.8}
$$

Furthermore, it is evident from (3.6.8) that

$$
\Re(p) > 0 \quad \Longleftrightarrow \quad \Re(G_0(\kappa, p)) > 0 \quad \forall \kappa \in \mathbb{R}.
\tag{3.6.9}
$$

This result shows that there cannot exist a root p in $\mathbb{C}$ with $\Re(p) > 0$ to $(3.6.4)_8$ unless $v_* < 0$. Since $v_* > 0$ it is clear that, at present, there cannot exist a p in $\mathbb{C}$ to $(3.6.4)_8$ with $\Re(p) > 0$ for any κ in $\mathbb{R} \setminus \{0\}$. Hence, when $v_0 = 0$ and inertial effects are disregarded the base process described in Section 3.1 is linearly stable with respect to all perturbations within the narrowed set under consideration.

If, in place of the foregoing normal mode analysis, a full-fledged Fourier-Laplace transform analysis of (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and

(3.5.12) is performed, then the narrowing of the class of initial data necessitated by the normal mode analysis does not occur. Furthermore, in this case it transpires that the Fourier-Laplace transform of s can be expressed in the form

$$S(\kappa, p) = \frac{\hat{h}(\kappa) + \frac{\gamma_l - \gamma_r}{v_*} H(\kappa, p)}{p + \frac{1}{v_*} F(\kappa, p)} \quad \forall (\kappa, p) \in I\!\!R \times \mathcal{C}, \qquad (3.6.10)$$

where $\hat{h}$ is the Fourier transform of h and, for each (κ, p) in $I\!\!R \times \mathcal{C}$, $H(\kappa, p)$ is a functional of the initial data η, ϖ and ϕ. From the foregoing discussion it is apparent that, since $v_* > 0$ at present, there exist no unstable zeros of the denominator of the expression on the right-hand-side of (3.6.10). Hence, when the base process is one wherein the associated phase boundary is static prior to the instant at which the perturbation is imposed and inertial effects are ignored, it is linearly stable with respect to all perturbations within the class introduced in Section 3.2.

3.7. Energy analysis for a base process involving a static interface with inertial effects present. Suppose, as in Section 3.6, that v_0 in (3.1.1) is zero; the parameter v_* is, as such, positive. Consider, now, the inertial initial boundary value problem formed by (3.5.2), (3.5.3) and (3.5.6)–(3.5.8). Observe that, since $v_0 = 0$, $(3.5.3)_2$ and (3.5.6) are replaced by $(3.6.1)_1$ and $(3.6.1)_2$, respectively. Furthermore, $(3.5.3)_1$ simplifies to its inertia-free counterpart (3.5.10). In place of a normal mode analysis like that performed in Section 3.6 an energy analysis will be used in this section to show that, when inertial effects are accounted for but $v_0 = 0$, the base process described by (3.1.1)–(3.1.3) is linearly stable with respect to all perturbations of the type put forth in Section 3.2. Preliminary to doing so define the *total energy* $\mathcal{E} : [0, t_*) \to I\!\!R_+$ by

$$\mathcal{E}(t) = E_K(t) + E_W(t) + E_T(t) \quad \forall t \in [0, t_*), \qquad (3.7.1)$$

where $E_K : [0, t_*) \to \bar{I\!R}_+$ is the *kinetic energy* given by

$$E_K(t) = \frac{\rho \ell}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \dot{w}(x_1, x_2, t)\, dx_1 dx_2 \quad \forall t \in [0, t_*), \qquad (3.7.2)$$

$E_W : [0, t_*) \to \bar{I\!R}_+$ is the *elastic energy* defined via

$$E_W(t) = \frac{\rho \ell}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{0} \left(a_l^2 w_{,1}^2 (x_1, x_2, t) + b_l^2 w_{,2}^2 (x_1, x_2, t) \right) dx_1 dx_2$$

$$+ \frac{\rho \ell}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{+\infty} \left(a_r^2 w_{,1}^2 (x_1, x_2, t) + b_r^2 w_{,2}^2 (x_1, x_2, t) \right) dx_1 dx_2$$

$$\forall t \in [0, t_*), \qquad (3.7.3)$$

and $E_T : [0, t_*) \to \bar{I\!R}_+$ is the *thermal energy* given by

$$E_T(t) = \frac{\rho \ell}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{0} c_l T^2(x_1, x_2, t) dx_1 dx_2 + \frac{\rho \ell}{2} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{0} c_r T^2(x_1, x_2, t) dx_1 dx_2$$

$$+ \rho \ell \int\limits_{0}^{t} \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{+\infty} k_l T_{,\alpha} (x_1, x_2, \tau) T_{,\alpha} (x_1, x_2, \tau) dx_1 dx_2 d\tau$$

$$+ \rho \ell \int\limits_{0}^{t} \int\limits_{-\infty}^{+\infty} \int\limits_{0}^{+\infty} k_r T_{,\alpha} (x_1, x_2, \tau) T_{,\alpha} (x_1, x_2, \tau) dx_1 dx_2 d\tau$$

$$\forall t \in [0, t_*). \qquad (3.7.4)$$

The constant ℓ which appears in (3.7.2)–(3.7.4) is assumed to be positive and carries units of length. It is clear from $(3.5.7)_{1,2,3}$ and the stipulated square integrability of $\eta_{,\alpha}$, ϖ and ϕ that $\mathcal{E}(0)$ exists. In writing (3.7.2)–(3.7.4) it is assumed, however, that there exists a positive time t_*, which may possibly be very small, such that the relevant integrals exist on $[0, t_*)$. A reasonable definition of

linear stability is, at present, that $\mathcal{E}$ remain bounded on $\mathbb{R}_+$. A straightforward but long calculation which makes use of (3.5.8), (3.6.1) and, recalling the foregoing remarks regarding the coincidence of $(3.5.3)_1$ with (3.5.10) when $v_0 = 0$, show that the *power* $\dot{\mathcal{E}}$ is given by

$$\dot{\mathcal{E}}(t) = -\rho \ell v_* \int\limits_{-\infty}^{+\infty} \dot{s}^2(x_2, t) \, dx_2 \quad \forall t \in [0, t_*). \tag{3.7.5}$$

Since $v_0 = 0$ at present, $\dot{\mathcal{E}}(t) \leq 0$ for all t in $[0, t_*)$; under these circumstances the interval over which $\mathcal{E}$ is defined can be extended incrementally to $\mathbb{R}_+$ leading to the following inequality:

$$\dot{\mathcal{E}}(t) \leq 0 \quad \forall t \in \mathbb{R}_+. \tag{3.7.6}$$

Evidently, then, by the definition of linear stability given above, the base process at hand is stable with respect to all perturbations under consideration if the associated phase boundary is static prior to the instant at which the perturbation is imposed and inertial effects are accounted for.

Note that a normal mode analysis akin to that performed in Section 3.6 produces the following dispersion relation

$$p + \frac{1}{v_*}(F_0(\kappa, p) + p G_0(\kappa, p)) = 0, \tag{3.7.7}$$

where $F_0 : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is defined by

$$F_0(\kappa, p) = \frac{a_l b_l a_r b_r (\gamma_l - \gamma_r)^2 \sqrt{\kappa^2 + p^2/b_l^2} \sqrt{\kappa^2 + p^2/b_r^2}}{a_l b_l \sqrt{\kappa^2 + p^2/b_l^2} + a_r b_r \sqrt{\kappa^2 + p^2/b_r^2}} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \tag{3.7.8}$$

and G_0 is as defined in (3.6.5). A study of this dispersion relation allows the recovery of the results obtained by the foregoing energy analysis.

The combined results of this and the preceeding section are consistent with those presented by FRIED [9] in the purely mechanical analogue of the problem

considered here. Hence, when $v_0 = 0$, the inclusion of thermal effects does not alter the linear stability of the base state (3.1.1)–(3.1.3) from its obvious mechanical analogue.

3.8. Normal mode analysis for a base process involving a moving interface with or without inertial effects. Suppose that v_0 in (3.1.1) is positive. Consider now both the inertia-free intial value problem consisting of (3.5.9), (3.5.2)$_{3,4}$, (3.5.10), (3.5.11), (3.5.3)$_{3,4}$, (3.5.11), (3.5.7)$_{3,4}$ and (3.5.8) and the inertial initial value problem comprised by (3.5.2), (3.5.3) and (3.5.6)–(3.5.8). Note that, in both of these cases, (3.1.5)$_3$, (3.1.11) and the assumed positivity of v_0 imply that $\ell_0 = 0$. Hence, in the inertia-free case, (3.5.11) and (3.5.12) simplify as shown below

$$[kT_{,1}] + \rho v_0 [c]T = \rho v_0 (\gamma_r - \gamma_l)\langle\!\langle a^2 w_{,1}\rangle\!\rangle \quad \text{on} \quad I,$$
$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*}\langle\!\langle a^2 w_{,1}\rangle\!\rangle + v_\vartheta T \quad \text{on} \quad I, \tag{3.8.1}$$

while, in the inertial case, (3.5.3)$_2$ and (3.5.6) become

$$[kT_{,1}] + \rho v_0 [c]T = \rho v_0 (\gamma_r - \gamma_l)\langle\!\langle (a^2 - v_0^2)w_{,1}\rangle\!\rangle \quad \text{on} \quad I,$$
$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*}\langle\!\langle (a^2 - v_0^2)w_{,1}\rangle\!\rangle + v_\vartheta T \quad \text{on} \quad I. \tag{3.8.2}$$

Next, a normal mode analysis analogous to those undertaken in Section 3.6 and [9] will be performed based upon the following representation of a solution to the relevant partial differential equations, jump conditions and kinetic relation:

$$w(x_1, x_2, t) = \begin{cases} W_l e^{+(\xi_l - v_0 t)x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^l, \quad t \in \mathbb{R}_+, \\ W_r e^{-(\xi_r - v_0 t)x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^r, \quad t \in \mathbb{R}_+, \end{cases}$$

$$T(x_1, x_2, t) = \begin{cases} \Theta_l e^{+(\zeta_l - v_0 t)x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^l, \quad t \in \mathbb{R}_+, \\ \Theta_r e^{-(\zeta_r - v_0 t)x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^r, \quad t \in \mathbb{R}_+, \end{cases} \tag{3.8.3}$$

$$s(x_2, t) = S e^{i\kappa x_2} e^{pt} \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Substitution of (3.8.3) into the equations appropriate to the equations appropriate to the inertia-free and inertial cases gives, respectively, the following dispersion relations

$$p + \frac{\nu^2 |\kappa|}{v_*}(1 + v_\vartheta G(\kappa, p)) = 0,$$

$$p + \frac{F(\kappa, p)}{v_*}(1 + v_\vartheta G(\kappa, p)) = 0,$$

(3.8.4)

where $G : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $F : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are given by

$$G(\kappa, p) = \frac{2v_*}{c_l - c_r + c_l g_l(\kappa, p) + c_r g_r(\kappa, p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},$$

$$F(\kappa, p) = \frac{(\gamma_l - \gamma_r)^2 (f_l(\kappa, p) f_r(\kappa, p) + v_0^2 p^2)}{f_l(\kappa, p) + f_r(\kappa, p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},$$

(3.8.5)

where $g_l : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $g_r : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are given by

$$g_l(\kappa, p) = \sqrt{1 + \frac{4}{v_0^2}(\alpha_l^2 \kappa^2 + \alpha_l p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},$$

$$g_r(\kappa, p) = \sqrt{1 + \frac{4}{v_0^2}(\alpha_r^2 \kappa^2 + \alpha_r p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},$$

(3.8.6)

and $f_l : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $f_r : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are defined via

$$f_l(\kappa, p) = \sqrt{(a_l^2 - v_0^2) b_l^2 \kappa^2 + a_l^2 p^2} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},$$

$$f_r(\kappa, p) = \sqrt{(a_r^2 - v_0^2) b_r^2 \kappa^2 + a_r^2 p^2} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}.$$

(3.8.7)

When $v_\vartheta = 0$ the dispersion relations in (3.8.4) reduce to

$$p + \frac{\nu^2 |\kappa|}{v_*} = 0,$$

$$p + \frac{F(\kappa, p)}{v_*} = 0,$$

(3.8.8)

Observe that $(3.8.8)_1$ and $(3.8.8)_2$ are structurally identical to the inertia-free and inertial dispersion relations obtained by FRIED [9] in the purely mechanical analogue of the investigation at hand. Hence, if the kinetic response function $\tilde{V}$

or $\tilde{\varphi}$ is chosen so that $v_\vartheta = 0$ then the linear stability of the base process at hand remains unaltered from that of its purely mechanical analogue by the presence of thermal effects; specifically, when $v_\vartheta = 0$, the linear stability of the base process (3.1.1)–(3.1.3) is determined entirely by the sign of v_*. That is, $v_* < 0$ is a necessary condition for the base process to be linearly unstable with respect to any perturbation of the type introduced in Section 3.2 and, further, $v_* < 0$ is a sufficient condition for the base process to be linearly unstable with respect to all but a small class of very special initial disturbances contained within the full set under consideration. Note, in particular, that $v_\vartheta = 0$ if either $\tilde{V}$ depends only on θ through the ratio f/θ or $\tilde{\varphi}$ is independent of θ. Assume, henceforth, that $v_\vartheta \neq 0$.

The branches of the square roots which define g_l and g_r are chosen so that, for p in $I\!\!R$,

$$1 + \tfrac{4}{v_0^2}(\alpha_l^2 \kappa^2 + \alpha_l p) > 0 \quad \Longrightarrow \quad g_l(\kappa, p) > 0 \quad \forall \kappa \in I\!\!R,$$
$$1 + \tfrac{4}{v_0^2}(\alpha_r^2 \kappa^2 + \alpha_r p) > 0 \quad \Longrightarrow \quad g_r(\kappa, p) > 0 \quad \forall \kappa \in I\!\!R, \tag{3.8.9}$$

from which it is clear that, for p in $\mathbb{C}$,

$$\Re(1 + \tfrac{4}{v_0^2}(\alpha_l^2 \kappa^2 + \alpha_l p)) > 0 \quad \Longrightarrow \quad \Re(g_l(\kappa, p)) > 0 \quad \forall \kappa \in I\!\!R,$$
$$\Re(1 + \tfrac{4}{v_0^2}(\alpha_r^2 \kappa^2 + \alpha_r p)) > 0 \quad \Longrightarrow \quad \Re(g_r(\kappa, p)) > 0 \quad \forall \kappa \in I\!\!R. \tag{3.8.10}$$

Evidently, then, (3.8.10) and (3.8.5)$_1$ yield the following result:

$$\Re(p) > 0 \quad \Longleftrightarrow \quad \Re(v_* G(\kappa, p)) > 0 \quad \forall \kappa \in I\!\!R. \tag{3.8.11}$$

The square roots which appear in f_l and f_r are defined via the principal branch of the complex logarithm. It is, therefore, clear that

$$\Re(p) > 0 \quad \Longleftrightarrow \quad \Re(F(\kappa, p)) > 0 \quad \forall \kappa \in I\!\!R. \tag{3.8.12}$$

An immediate consequence of (3.8.11) and (3.8.12) is that one or both of the parameters v_* or v_ϑ must be negative in order for a root p in $\mathbb{C}$ of either $(3.8.4)_1$ or $(3.8.4)_2$ to have a positive real part. It is also obvious, from (3.8.11) and (3.8.12), that if both $v_* < 0$ and $v_\vartheta < 0$ then there exists a root p in $\mathbb{C}$ with $\Re(p) > 0$ to both of (3.8.4) regardless of the value of the wave number κ in $\mathbb{R} \setminus \{0\}$. A more subtle condition sufficient for the existence of a root p in $\mathbb{C}$ to either of (3.8.4) occurs under the assumption that $v_* > 0$, $v_\vartheta < 0$ and $v_*|v_\vartheta|/c_l > 1$. Specifically, when $v_* > 0$, $v_\vartheta < 0$ and $v_*|v_\vartheta|/c_l > 1$ it is, then, possible to show that there exists a root p in $\mathbb{C}$ to both of (3.8.4) provided the wave-number κ in $\mathbb{R} \setminus \{0\}$ is sufficiently small so that the inequality

$$\frac{c_l - c_r}{c_l + c_r} + \frac{f_l(\kappa, p)}{1 + \frac{c_r}{c_l}} + \frac{f_r(\kappa, p)}{1 + \frac{c_l}{c_r}} < \frac{2v_*|v_\vartheta|}{c_l + c_r} \qquad (3.8.13)$$

holds. A similar condition which guarantess the existence of a root p in $\mathbb{C}$ to either of (3.8.4) occurs under the assumption that $v_* < 0$, $v_\vartheta > 0$ and $|v_*|v_\vartheta/c_l < 1$. In this case there always exists such a root to either of (3.8.4) as long as the wave-number κ is sufficiently large so that the following inequality is satisfied:

$$\frac{c_l - c_r}{c_l + c_r} + \frac{f_l(\kappa, p)}{1 + \frac{c_r}{c_l}} + \frac{f_r(\kappa, p)}{1 + \frac{c_l}{c_r}} > \frac{2|v_*|v_\vartheta}{c_l + c_r} \qquad (3.8.14)$$

The foregoing discussion shows that, unlike the purely mechanical process investigated in [9], the present context is not, when $v_0 > 0$, amenable to the statement of necessary and sufficient conditions for the linear instability of the base process at hand. The sufficient conditions which have been presented above are, however, of interest.

3.9 Conclusion. In [9] it is demonstrated that when the purely mechanical analogue of the parameter v_* is positive the appropriate purely mechanical version of the base process considered here is linearly stable with respect to all perturbations which are considered in that context. The last of the conditions sufficient

for the linear instability of the thermoelastic base process (3.1.1)–(3.1.3), *viz.*,

$$v_* > 0, \quad v_\vartheta < 0, \quad \frac{v_* v_\vartheta}{c_l} > 1, \quad \frac{c_l - c_r}{c_l + c_r} + \frac{f_l(\kappa, p)}{1 + \frac{c_r}{c_l}} + \frac{f_r(\kappa, p)}{1 + \frac{c_l}{c_r}} < \frac{2v_* |v_\vartheta|}{c_l + c_r}, \quad (3.9.1)$$

where the parameters v_*, v_ϑ, c_l and c_r are as defined in $(3.4.10)_1$ or $(3.4.11)_1$, $(3.4.10)_2$ or $(3.4.11)_2$, $(3.5.5)_4$ and $(3.5.5)_3$, respectively, and the functions f_l and f_r are as defined in (3.8.7), is, hence, arguably the most interesting of the three which are presented. It exposes what might be described as a competition between mechanically *stabilizing* and thermally *destabilizing* effects and an explicit dependence of growth-rate upon wave-number. Significantly in these circumstances, it is the low wave-numbers (that is, *long waves*) with respect to which the base process is linearly unstable. Under conditions consistent with (3.9.1), a moving planar phase boundary, therefore, tends to prefer a highly wrinkled—*i.e.*, plate-like or dendritic—morphology. Instability of this variety is also found in models for dendritic crystal growth and solidification (see [13–14] and [16]).

In analogy to [9] where the physical plausibility of a purely mechanical kinetic response function for which the parameter analogous to v_* can be negative is addressed, it is now natural to consider the question of whether it is physically reasonable for a kinetic response function to depend monotonically on its first argument—so that v_* is always positive and the related purely mechanical process is linearly stable—but non-monotonically on its second argument, in which case v_ϑ may be negtive. The experimental work of CLAPP & YU [5] which studies, in part, the dependence upon temperature of transformation kinetics in a particular alloy capable of sustaining displacive solid-solid phase transformations indicates that the role of temperature in such kinetics is very complicated. In fact, despite what appears to be a very careful experimental procedure and analysis, CLAPP & YU [5] observe a severe scatter in the data which measure the dependence of phase boundary velocity upon temperature. This scatter indicates there may not be a simple functional dependence of interface normal velocity upon temperature. With regard to the issue at hand, these experimental results seem to indicate that,

if a kinetic relation of the form (2.5.1) or (2.5.2) is insisted upon, monotonicity of a kinetic response function $\tilde{V}(\Phi, \cdot)$ for fixed Φ in $\mathbb{R}$ or $\tilde{\varphi}(V, \cdot)$ for fixed V in $\mathbb{R}$ may be the exception rather than the rule.

ACKNOWLEDGEMENTS

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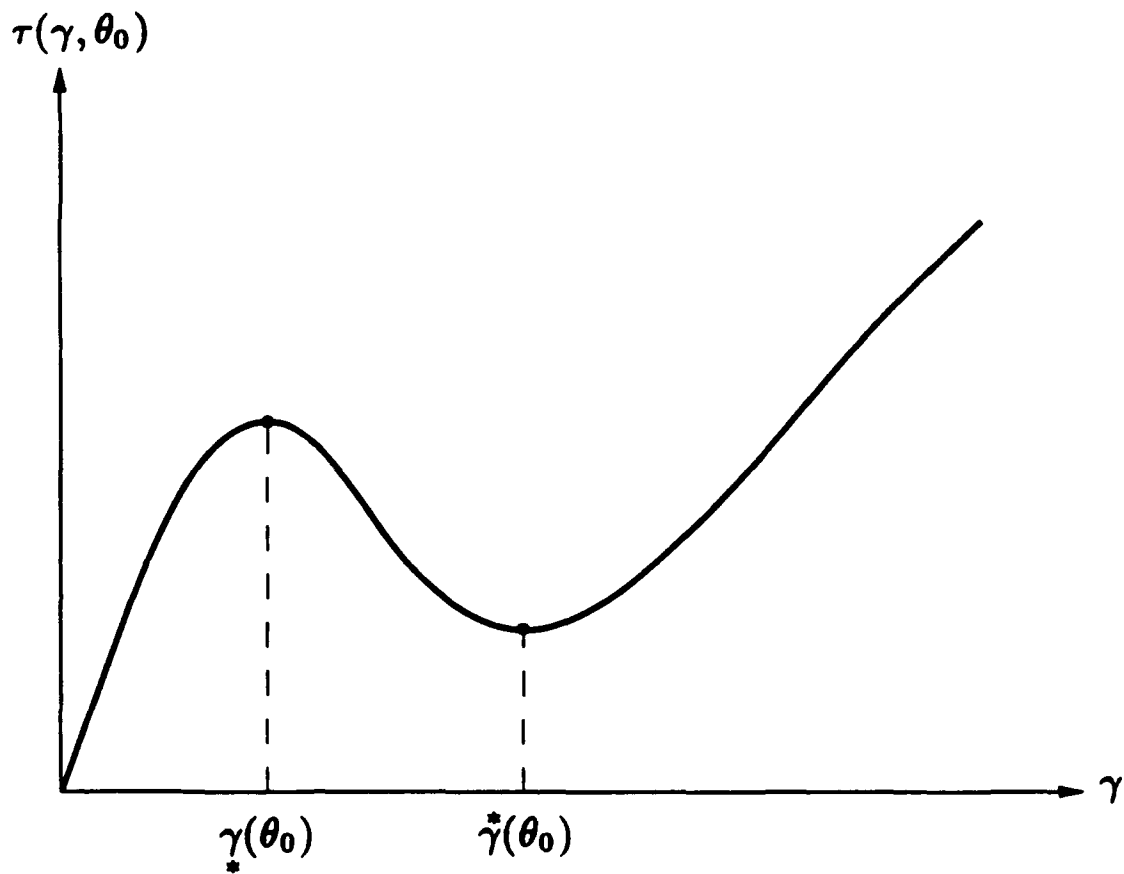[16] J. STRAIN, Linear stability of planar solidification fronts, *Physica D* **30** (1988), 277–320.

Figure 1: Graph of the shear stress response function $\tau(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M).
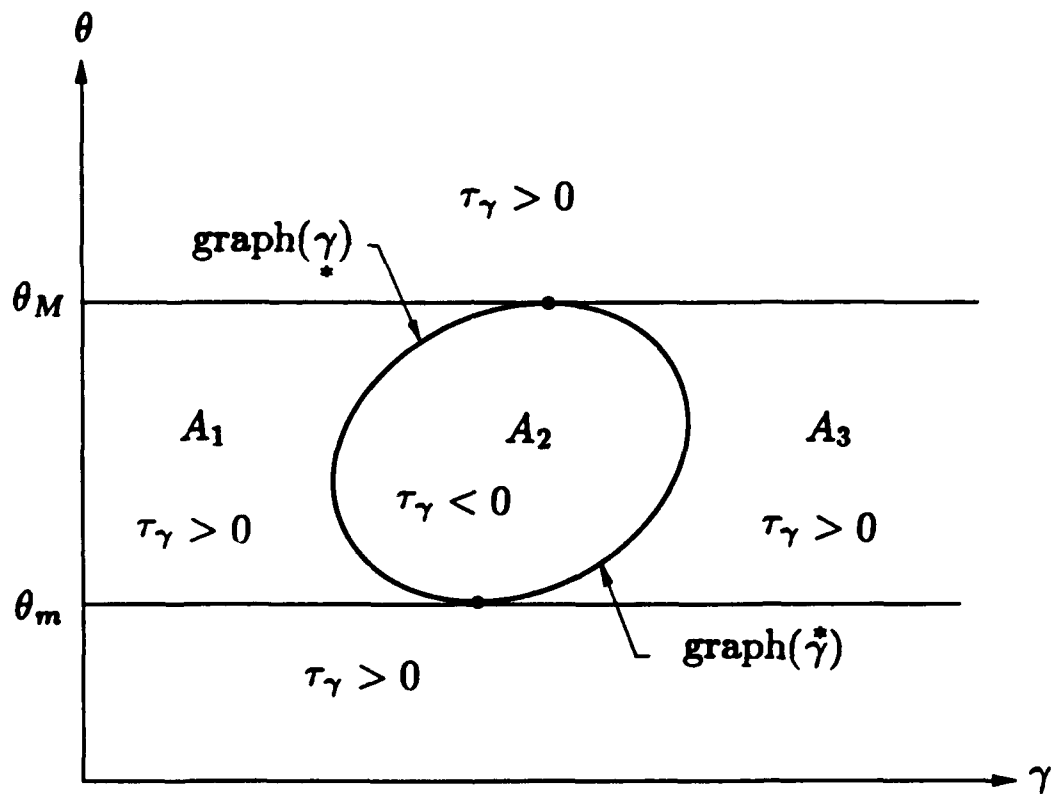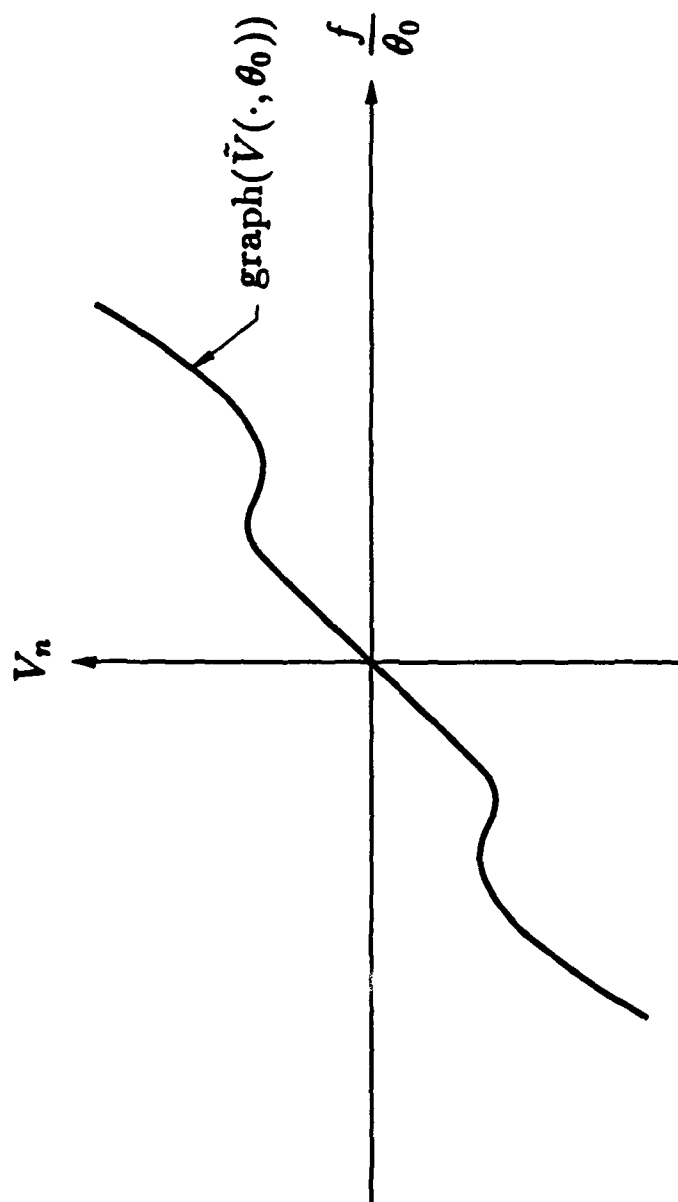
Figure 2: Plot of the shear strain-temperature quadrant.

Figure 3: Graph of the kinetic response function $\tilde{V}(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M).
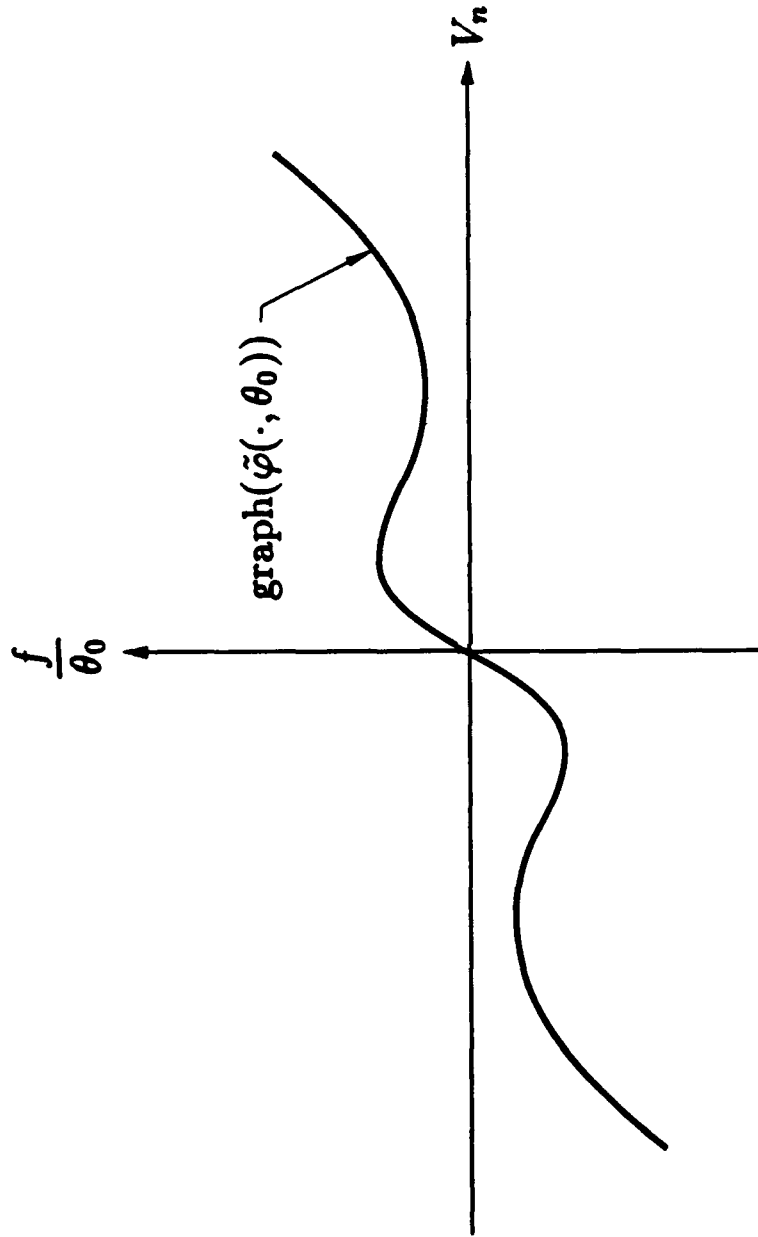
V_n

$\dfrac{f}{\theta_0}$

graph($\tilde{\varphi}(\cdot,\theta_0)$))

Figure 4: Graph of the kinetic response function $\tilde{\varphi}(\cdot,\theta_0)$ for fixed θ_0 in (θ_m,θ_M).

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This investigation is directed toward understanding the role of coupled mechanical and thermal effects in the linear stability of an isothermal antiplane shear motion which involves a single planar phase boundary in a non-elliptic thermoelastic material which has multiple elliptic phases. When the relevant process is static -- so that the phase boundary does not move prior to the imposition of the disturbance -- it is shown to be linearly stable. However, when the process involves a steadily propagating phase boundary it may be linearly unstable. Various conditions sufficient to guaranted the linear instability of the process are obtained. These conditions depend on the monotonicity of the kinetic response function -- a constitutively supplied entity which relates the driving traction acting on a phase boundary to the local absolute temperature and the normal velocity of the phase boundary -- and, in certain cases, on the spectrum of wave-numbers associated with the perturbation to which the process is subjected. Inertia is found to play an insignificant role in the qualitative features of the aforementioned

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#19. sufficient conditions. It is shown, in particular, that instability can arise even when the normal velocity of the phase boundary is an increasing function of the driving traction if the temperature dependence in the kinetic response function is of a suitable nature. Significantly, the instability which is present in this setting occurs only in the long waves of the Fourier decomposition of the moving phase boundary, implying that the interface prefers to be highly wrinkled.